Evolution algebras are finitely universal and Cayley evolution algebras

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Finitely universal

Definition

Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. If every finite group G can be represented as $\mathrm{Aut}_{\mathcal{C}}(X) \cong G$ for some $X \in \mathrm{Obj}(\mathcal{C})$ we say that $\mathcal{C}$ is *finitely universal* (f.u.).

Finitely universal

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Identifying f.u. categories is a hard problem:

- $\mathcal{G}roups$ is not f.u. (there does not exist $O \in \mathrm{Obj}(\mathcal{G}\mathrm{roups})$ such that $\mathrm{Aut}(O) \cong \mathbb{Z}_3$).
- $\mathcal{A}lg_{\Bbbk}$ (unital commutative associative $\Bbbk$-algebras) is f.u.
- $\mathcal{G}raphs$ (finite simple non oriented) is f.u. (Frucht 39')

Motivation for evolution algebras

- Evolution algebras model self-reproduction in non-Mendelian genetics.
- Example of Mendelian genetics: E and e two alleles with multiplication table of the gametic algebra given by

	E	e
E	E	$\frac{1}{2}(E+e)$
e	$\frac{1}{2}(E+e)$	e

- Example of non-Mendelian genetics: a_1 and a_2 two alleles with multiplication table of the gametic algebra given by

	a_1	a_2
a_1	$\omega_{11}a_1+\omega_{21}a_2$	0
a_2	0	$\omega_{12}a_1+\omega_{22}a_2$

with $\omega_{11}+\omega_{21}=\omega_{12}+\omega_{22}=1$.

Basics on evolution algebras

Definition

An *evolution algebra* (e.a.) over a field $\mathbb{k}$ is a $\mathbb{k}$-algebra X provided with a basis $B = \{b_i \mid i \in \Lambda\}$ such that $b_i b_j = 0$ whenever $i \neq j$. Such a basis B is called a *natural basis*.

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Definition

Let X be an e.a. with natural basis B. The scalars $\omega_{ki} \in \Bbbk$ such that

$$b_i^2 := b_i b_i = \Sigma_{k \in \Lambda} \omega_{ki} b_k$$

are called the *structure constants* of X relative to B, and the matrix $M_B := (\omega_{ki})$ is said to be the *structure matrix* of X relative to B.

We call $\mathcal{E}_{\Bbbk}$ the category of evolution algebras over a field $\Bbbk$.

Evolution algebras and graphs

Definition

Let X be an n-dimensional e.a. over $\Bbbk$ with natural basis B and structure matrix M_B. The directed graph $\Gamma(X, B) = (V, E)$ with $V = \{1, \ldots, n\}$, $E = \{(j, i) \in V \times V : \omega_{ij} \neq 0\}$ is called the *oriented graph attached to the e.a. X relative to B.*

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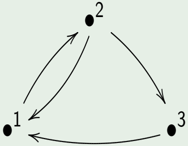
Example

Consider A an evolution algebra with natural basis $B = \{b_1, b_2, b_3\}$ such that $b_1^2 = 2b_2$, $b_2^2 = b_1 + b_3$, $b_3^2 = b_1$.

So, the structure matrix is

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and the directed graph attached to A is



Graphs and evolution algebras

We define a covariant faithful functor $\mathcal{X}\colon \mathcal{S}Graphs \rightsquigarrow \mathcal{E}_{\Bbbk}$ such that:

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- for every $\mathcal{G} = (V, E) \in \mathcal{S}Graphs$ we define $\mathcal{X}(\mathcal{G})$ to be the e.a. over $\mathbb{k}$, with natural basis

$$B = \{b_v : v \in V\} \cup \{b_e : e \in E\}$$

and multiplication given by $b_v^2 = b_v$ for all $v \in V$ and $b_e^2 = b_e + \Sigma_{v \in e} b_v$ for all $e \in E$.

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- for every morphism $f : \mathcal{G}_1 \to \mathcal{G}_2$, we have an *injective* morphism $f : V_1 \to V_2$ such that if $e = \{v, w\} \in E_1$ then $f(e) = \{f(v), f(w)\} \in E_2$.

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  So, we define $\mathcal{X}(f) : \mathcal{X}(\mathcal{G}_1) \to \mathcal{X}(\mathcal{G}_2)$ with $\mathcal{X}(f)(b_v) = b_{f(v)}$ and $\mathcal{X}(f)(b_e) = b_{f(e)}$.

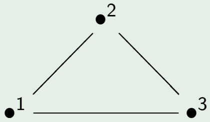
Example

$$\mathcal{X}(G),\ B = \{b_v : v \in V\} \cup \{b_e : e \in E\}$$

$$b_v^2 = b_v \text{ for all } v \in V \text{ and } b_e^2 = b_e + \Sigma_{v \in e} b_v \text{ for all } e \in E$$

Example

Consider the simple non oriented graph $\mathcal{G}$.
We define the natural basis of $\mathcal{X}(\mathcal{G})$ as
$B = \{b_1,\ b_2,\ b_3,\ b_{12},\ b_{23},\ b_{13}\}$ with
multiplication $b_1^2 = b_1,\ b_2^2 = b_2,\ b_3^2 = b_3,$



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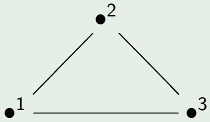

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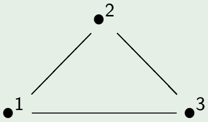
$b_{12}^2 = b_1 + b_2 + b_{12}$, $b_{23}^2 = b_2 + b_3 + b_{23}$,
$b_{13}^2 = b_1 + b_3 + b_{13}$.



The structure matrix of $\mathcal{X}(G)$ is $\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

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Main result

Theorem

Let $\mathcal{E}_{\Bbbk}$ be the category of evolution algebras over a field $\Bbbk$ and G be a finite group. Then there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_{\Bbbk})$ such that X is regular and $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X) \cong G$. In particular the category $\mathcal{E}_{\Bbbk}$ is finitely universal.

Definition

An e.a. X is *regular* if and only if $X = X^2$, or equivalently, if and only if for any natural basis B, the structure matrix M_B is a regular matrix ($\det(M_B) \neq 0$).

Sketch of the proof

Theorem

Let $\mathcal{E}_{\Bbbk}$ be the category of evolution algebras over a field $\Bbbk$ and G be a finite group. Then there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_{\Bbbk})$ such that X is regular and $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X) \cong G$.

- Given a finite group G there exists a simple non oriented graph $\mathcal{G}$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$.
- Using the covariant faithful functor $\mathcal{X}$ we get that $\mathcal{X}(\mathcal{G})$ is regular.
- **$\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathcal{X}(\mathcal{G})) \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$?**

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- Given a finite group G there exists a simple non oriented graph $\mathcal{G}$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$.
- Using the covariant faithful functor $\mathcal{X}$ we get that $\mathcal{X}(\mathcal{G})$ is regular.
- $\mathbf{Aut}_{\mathcal{E}_{\Bbbk}}\left(\mathcal{X}(\mathcal{G})\right) \cong \mathbf{Aut}_{\mathcal{G}raphs}(\mathcal{G})$**?**
- Given a simple graph $\mathcal{G} = (V, E)$, and $g \in \mathrm{Aut}_{\mathcal{E}_{\Bbbk}}\left(\mathcal{X}(\mathcal{G})\right)$, there exists $f \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ such that $g = \mathcal{X}(f)$.
- Given G a finite group, then there exists an e.a. X such that X is regular and $\mathrm{Aut}(X) \cong G$.

Finite algebraic affine group scheme

Any e.a. X over $\Bbbk$ defines an algebraic affine group scheme over $\Bbbk$, $\mathrm{Aut}_{\mathcal{E}_\Bbbk}(X)\colon \mathcal{A}lg_\Bbbk \rightsquigarrow \mathcal{G}roups$, that takes any $R \in \mathrm{Obj}(\mathcal{A}lg_\Bbbk)$ to $\mathrm{Aut}_{\mathcal{E}_\Bbbk}(X_R)$, the group of R-algebra automorphisms of $X_R = X \otimes_\Bbbk R$.

Theorem

Given any field $\Bbbk$, and any constant finite algebraic affine group scheme $\mathbf{G}$ *over $\Bbbk$, there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_\Bbbk)$ such that the group scheme $\mathrm{Aut}_{\mathcal{E}_\Bbbk}(X)$ is isomorphic to* $\mathbf{G}$.

Finite algebraic affine group scheme

Theorem

Given any field $\mathbb{k}$, and any constant finite algebraic affine group scheme $\mathbf{G}$ *over $\mathbb{k}$, there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_{\mathbb{k}})$ such that the group scheme $\mathrm{Aut}_{\mathcal{E}_{\mathbb{k}}}(X)$ is isomorphic to* $\mathbf{G}$.

Main idea. $G = \mathbf{G}(\mathbb{k})$, $\exists\ \mathcal{G}$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$. There exists an exact sequence of algebraic affine group schemes:

$$1 \to \mathbf{Diag}(\Gamma) \to \mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(\mathcal{X}(\mathcal{G})) \to \mathrm{Aut}(\Gamma)$$

$\Gamma = \Gamma(\mathcal{X}(\mathcal{G}), B)$ is the oriented graph attached to $\mathcal{X}(\mathcal{G})$, $\mathbf{Diag}(\Gamma)$ is the diagonal group scheme of Γ (that turns out to be 1), $\mathrm{Aut}(\Gamma)$ is the constant group scheme given by the group of oriented graph automorphisms of Γ and $\mathrm{Aut}_{\mathcal{E}_{\mathbb{k}}}(\mathcal{X}(\mathcal{G})) \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \mathbf{G}$.

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Cayley evolution algebras

Definition

Given a finite group G and a field $\mathbb{k}$ one can consider a set-theoretical map $f : G \to \mathbb{k}$. The *Cayley e.a. associated to f* is the algebra $\mathrm{Cay}(f)$ given by the $\mathbb{k}-$vectorial space, $\mathbb{k}[G]$, provided with a natural basis $B = \{g \mid g \in G\}$ such that $g \cdot h = 0$ whenever $g \neq h$ and $g \cdot g = \Sigma_{h \in G} f(h)gh$.

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Theorem

Let $\mathcal{E}_{\mathbb{k}}$ be the category of evolution algebras over a large enough field $\mathbb{k}$ and G be a finite group. Then there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_{\mathbb{k}})$ such that X is simple and $\mathrm{Aut}_{\mathcal{E}_{\mathbb{k}}}(X) \cong G$. In particular the category $\mathcal{E}_{\mathbb{k}}$ is finitely universal.

Definition

An e.a. X is *simple* if and only if $X^2 \neq 0$ and 0 is the only proper ideal.

Sketch of the proof

Theorem

Let $\mathcal{E}_{\mathbb{k}}$ be the category of evolution algebras over a large enough field $\mathbb{k}$ and G be a finite group. Then there are infinitely many non isomorphic $X \in \operatorname{Obj}(\mathcal{E}_{\mathbb{k}})$ such that X is simple and $\operatorname{Aut}_{\mathcal{E}_{\mathbb{k}}}(X) \cong G$.

- $G \leq \operatorname{Aut}_{\mathcal{E}_{\mathbb{k}}}(\operatorname{Cay}(f))$.

Sketch of the proof

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- $G \leq \mathrm{Aut}_{\mathcal{E}_{\mathbb{k}}}(\mathrm{Cay}(f))$.
- If $\mathrm{Cay}(f)$ is regular and $S = \mathrm{supp}(f)$ generates G, then $\mathrm{Cay}(f)$ is simple.

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- $G \leq \mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathrm{Cay}(f))$.
- If $\mathrm{Cay}(f)$ is regular and $S = \mathrm{supp}(f)$ generates G, then $\mathrm{Cay}(f)$ is simple.
- If $\mathrm{Cay}(f)$ is regular, $S = \mathrm{supp}(f)$ generates G, $f|_S$ injective and in S there are elements of coprime order, then $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathrm{Cay}(f)) \cong G$.

Sketch of the proof

Theorem

Let $\mathcal{E}_{\Bbbk}$ be the category of evolution algebras over a large enough field $\Bbbk$ and G be a finite group. Then there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_{\Bbbk})$ such that X is simple and $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X) \cong G$.

- $G \leq \mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathrm{Cay}(f))$.
- If $\mathrm{Cay}(f)$ is regular and $S = \mathrm{supp}(f)$ generates G, then $\mathrm{Cay}(f)$ is simple.
- If $\mathrm{Cay}(f)$ is regular, $S = \mathrm{supp}(f)$ generates G, $f|_S$ injective and in S there are elements of coprime order, then $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathrm{Cay}(f)) \cong G$.
- If $\Bbbk$ is a large enough field, given a finite group G and any $S \subset G$, there exists a set-theoretical map $f : G \to \Bbbk$ such that $S = \mathrm{supp}(f)$, $f|_S$ is injective and $\mathrm{Cay}(f)$ is regular.
- Given G a finite group, if $\Bbbk$ is large enough, then there exists an e.a. X such that X is simple and $\mathrm{Aut}(X) \cong G$.

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Thank you for your attention!