Celebrative talk.
Enrique's homemade Algebraic Geometry.

Celebrating Enrique Arrondo's 60th birthday

Alicia Tocino

July 12, 2023

Where does Enrique come from?

Projective Algebraic Geometry in Milano, June, 2009

Where does Enrique come from?

- 1985: Degree in Mathematical Science in the Universidad Complutense de Madrid (with extraordinary bachelor's degree award).

- 1986-1989: Predoctoral FPI grant from the MEC at the Faculty of Mathematical Sciences of the Universidad Complutense de Madrid.

- 1990: PhD in Mathematical Sciences from the Universidad Complutense de Madrid (with extraordinary PhD award).

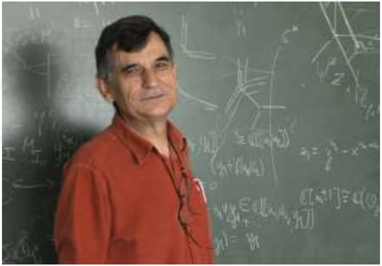
- Professor at Universidad Complutense de Madrid since 1990 until it holds.

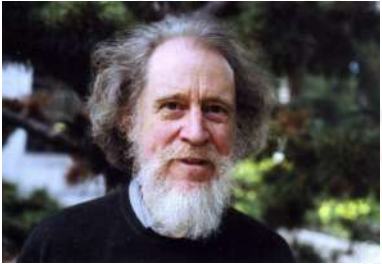
Where does Enrique come from?

1990 Enrique Arrondo Esteban, Universidad Complutense de Madrid

↓

1975 Ignacio Sols Lucía, Universidad de Zaragoza





Where does Enrique come from?

1975 Ignacio Sols Lucía, Universidad de Zaragoza

GESTA 2009, Benasque, Spain

Where does Enrique come from?

- **Ignacio Sols (1975)**
  - Rafael Hernández (1985) → María Jesús Vázquez (1997)
  - Raquel Mallavibarrena (1987)
    - Roberto Muñoz (1999)
    - Sonia González (2004)
  - Manuel Pedreira (1987)
    - Luis Solá (2002) → Alberto Franceschini (2023)
    - Luis Fuentes (2002)
    - Rosa Cid (2003)
  - **Enrique Arrondo (1990)**
    - **Raquel Díaz (1996)**
    - **Luca Ugaglia (2001)**
    - **Beatriz Graña (2003)**
    - **Jose Carlos Sierra (2004)**
    - **Jorge Caravantes (2006)**
    - **Sofía Cobo (2008)**
    - **Simone Marchesi (2012)**
      - **Aydee López (2017)**
      - **Aline Vilela (2018)**
    - **Alicia Tocino (2015)**
  - Luis Giraldo (1993)
    - Alvaro Bustinduy (2004)
    - Carlos Pérez (2014)
  - Francisco Presas (2000)
    - Roger Casals (2015) → James Hughes (2023)
    - Álvaro del Pino (2017) → L. Toussaint (2020)
    - Jose Luis Pérez (2019)
    - Eduardo Fernández (2021)
  - Carlos Hermoso (2002)

Where does Enrique come from?

1975 Ignacio Sols Lucía,
Universidad de Zaragoza

↓

1960 José Luis Viviente Mateu,
Universidad Complutense de Madrid

1960 José Luis Viviente Mateu,
Universidad Complutense de Madrid

↓

1941 Pedro Abellanas Cebollero,
Universidad de Madrid

Mathematics Genealogy Project https://www.genealogy.math.ndsu.nodak.edu/

Where does Enrique come from?

1941 Pedro Abellanas Cebollero,
Universidad de Madrid

↓

1935 Tomás Rodríguez Bachiller,
Universidad de Madrid

1935 Tomás Rodríguez Bachiller,
Universidad de Madrid

↓

1901 José María Plans y Freyre
José Gabriel Álvarez Ude,
Universidad de Madrid

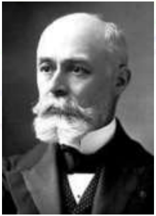
Mathematics Genealogy Project https://www.genealogy.math.ndsu.nodak.edu/

Where does Enrique come from?

José Gabriel Álvarez Ude,
Universidad de Madrid

↓

1873 Eduardo Torroja Caballé,
Universidad de Madrid



1873 Eduardo Torroja Caballé,
Universidad de Madrid

↓

1822 Karl Georg Christian von Staudt,
Friedrich-Alexander-Universität Erlangen-Nürnberg

Mathematics Genealogy Project https://www.genealogy.math.ndsu.nodak.edu/

Where does Enrique come from?

1822 Karl Georg Christian von Staudt

↓

1799 Carl Friedrich Gauss, Universität Helmstedt

⋮

Mathematics Genealogy Project https://www.genealogy.math.ndsu.nodak.edu/

Congruences with *Ignacio Sols*

- PhD title: Congruencias de rectas en $\mathbb{P}^3$ $\Rightarrow$ **On congruences of lines in the projective space**, with Ignacio Sols (1992).
  - Study of smooth congruences (surfaces in $\mathbb{G}(1,3)$) $\rightsquigarrow$ parallelism with surfaces in $\mathbb{P}^4$.
  - They proved that the only indecomposable bundles on $\mathbb{G}(1,3)$ without intermediate cohomology are line bundles and twists of the spinor bundle.
  - They described the Hilbert schemes of all smooth congruences of degree up to nine are described, enhancing a paper by the authors and a paper by *Alessandro Verra*.
  - Most original result $\rightsquigarrow$ five classes of the smooth congruences that can be obtained as a projection from another surface in $\mathbb{G}(1,4)$.
  - Finitely many components of the Hilbert scheme consisting of smooth congruences that are not of general type $\rightsquigarrow$ analogous result for surfaces in $\mathbb{P}^4$ was proven by *Geir Ellingsrud* and *Christian Peskine*.

In the meantime

- **Proof of Schubert's conjectures on double contacts**, with *Raquel Mallavibarrena* and *Ignacio Sols* (1990).

Facultad de Matemáticas, Universidad Complutense de Madrid, "1992"

Also in the meantime

Madrid, 199?

Congruences with *Mark Gross*

- **On smooth surfaces in $\mathbb{G}(1,3)$ with a fundamental curve** (1993).
  - $\mathbb{G}(1,3)$ contains two families of planes.
  - The bidegree of a surface in $\mathbb{G}(1,3)$ is determined by the number of points where the congruence intersects a general plane from each family.
  - A singular point of a congruence is a point in $\mathbb{P}^3$ with infinitely many lines of the congruence passing through it $\rightsquigarrow$ form the fundamental curve of the congruence.
  - More comprehensive classification of smooth congruences in $\mathbb{G}(1,3)$ (giving the possible bidegrees of the congruences when its fundamental curve is not a line and the degree of the fundamental curves).
  - Construction of smooth congruences with a bidegree of (3,6) and (5,8).
  - Generalization is coming...

Congruences with *Marina Bertolini* and *Cristina Turrini*

Europroj meeting "EuroConference, Noordfjiordeid, Norway, June 1995

Congruences with *Marina Bertolini* and *Cristina Turrini*

- First part ⇒ **Classification of smooth congruences with a fundamental curve** (1994).
  - A congruence of lines is a variety of dimension $n-1$ in $\mathbb{G}(1,n)$.
  - Classification all smooth congruences having a fundamental curve C.
    - C is a line ⇒ infinitely many.
    - $\deg(C) \geq 2$ ⇒ finitely many.
- Second part ⇒ **Congruences of small degree in** $\mathbb{G}(1,4)$ (1998).
  - Classification of all smooth threefolds in $\mathbb{G}(1,4)$ of bidegree (a,b) with $a=0$ or $b \leq 2$.
  - All possible numerical invariants of smooth threefolds in $\mathbb{G}(1,4)$ of degree ≤ 10.
  - Classification work of varieties of small degree neeed given by *Paltin Ionescu* (1988) and *Maria Lucia Fania* and *Elvira Laura Livorni* (1994).

More congruences with *M. Bertolini* and *C. Turrini*

- Third part $\Rightarrow$ **Quadric bundle congruences in** $\mathbb{G}(1,n)$ (2000).
  - All possible smooth congruences in $\mathbb{G}(1,n)$ for $n \geq 4$, which have a quadratic bundle structure over a curve.
  - Main tool: Castelnuovo's bound for the genus of projective curves & generalization for curves in an arbitrary Grassmannian variety (which was obtained by *Luis Giraldo*, 2000).
- More:
  - **Genus formula for generalized offset curves**, with *Juana Sendra* and *J. Rafael Sendra* (1999).
  - **A focus on focal surfaces**, with *M. Bertolini* and *C. Turrini* (2001)
  - **Line congruences of low order**, on his own (2002).
    - Classification of line congruences of order 0 and 1.
  - **Focal loci in** $\mathbb{G}(1,n)$, with *M. Bertolini* and *C. Turrini* (2005).
  - **On the ampleness of the normal bundle of line congruences**, with *M. Bertolini* and *C. Turrini* (2011).

Congruences with *Beatriz Graña* and *Sofía Cobo*

- **Congruences on $\mathbb{G}(1,4)$ with split universal quotient bundle**, with **Beatriz Graña** (2006).

Florence, Italy, March 2004

Congruences with *Beatriz Graña* and *Sofía Cobo*

- **Congruences on $\mathbb{G}(1,4)$ with split universal quotient bundle**, with *Beatriz Graña* (2006).
  - Classification of the smooth threefolds in $\mathbb{G}(1,4)$ for which the restriction of the universal quotient bundle $\mathcal{Q}$ is a direct sum of two line bundles.
- **On the stability of the universal quotient bundle restricted to congruences of low degree of $\mathbb{G}(1,3)$**, with **Sofía Cobo** (2010).
  - Is there a congruence with any preassigned bidegree (a,b)?

Vector bundles without intermediate cohomology

- *Geoffrey Horrocks* criterion (1964): a vector bundle F over $\mathbb{P}^n$ splits if and only if $H^i(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < i < n$.

- *Giorgio Ottaviani* criterion (1987): a vector bundle F over $\mathbb{G}(k, n)$ splits if and only if $H^i(\mathbb{G}(k, n), \wedge^{i_1}\mathcal{Q} \otimes \ldots \otimes \wedge^{i_s}\mathcal{Q} \otimes F(t)) = 0$ for all $0 \leq i_1, \ldots, i_s \leq n - k$, $s \leq k$, $t \in \mathbb{Z}$ and $0 < i < (k+1)(n-k)$ where $\mathcal{Q}$ is the quotient bundle on $\mathbb{G}(k, n)$.

- **Vector bundles on $\mathbb{G}(1, 4)$ without intermediate cohomology**, with *Beatriz Graña* (1999).
  - Characterization of vector bundles on $\mathbb{G}(1, 4)$ without intermediate cohomology.

- **Cohomological Characterization of Vector Bundles on Grassmannians of Lines**, with *Francesco Malaspina* (2010).
  - Following the notion of Mumford–Castelnuovo regularity.

Vector bundles without intermediate cohomology

- **Cohomological characterization of universal bundles of $\mathbb{G}(1, n)$,** with **Alicia Tocino** (2019).
  - Characterization of directs sums of twists of symmetric powers of the universal quotient bundle over $\mathbb{G}(1, n)$ (also using Beilinson's spectral sequence).

PhD defense, Universidad Complutense de Madrid, July 2015

Vector bundles without intermediate cohomology

- **Vector bundles on Fano 3-folds without intermediate cohomology**, with *Laura Costa* (2000).
  - There exist, up to a twist, only three indecomposable rank-2 bundles without intermediate cohomology.
- **Vector bundles with no intermediate cohomology on Fano threefolds of type V_{22}**, with *Daniele Faenzi* (2006).
  - Up to isomorphism and twisted by line bundles, the only possible rank-2 vector bundles without intermediate cohomology on prime Fano threefold of index 1 and genus 12 are of five different classes.
- **Curves and vector bundles on quartic threefolds**, with *Carlo G. Madonna* (2009).
  - Vector bundles without intermediate cohomology of rank ≥ 3 on hypersurfaces inside $\mathbb{P}^4$ of degree $r \geq 1$.

Projections of Grassmannians

- **Projections of Grassmannians of lines and characterization of Veronese varieties**, on his own (1999).
  - The only nondegenerate smooth complex surface in $\mathbb{P}^5$ that can be isomorphically projected to $\mathbb{P}^4$ is the Veronese surface (F. Severi).
  - For $n \geq 2$, the only nondegenerate n-dimensional smooth subvariety of $\mathbb{P}^{n(n+3)/2}$ that can be isomorphically projected to $\mathbb{P}^{2n}$ is the double Veronese embedding of $\mathbb{P}^n$ (F. L. Zak).
  - Characterization of the double Veronese embedding of $\mathbb{P}^n$ as the only variety that, under certain general conditions, can be isomorphically projected from the $\mathbb{G}(1, 2n+1)$ to $\mathbb{G}(1, n+1)$.
  - Next steps are coming.
- **The universal rank-$(n-1)$ bundle on $\mathbb{G}(1, n)$ restricted to subvarieties**, on his own (1998).
  - The same problem is studied for $\mathbb{G}(n-1, n)$ and $\mathbb{G}(n-2, n)$.

Projections of Grassmannians

- **Classification of *n*-dimensional subvarieties of $\mathbb{G}(1, 2n)$ that can be projected to** $\mathbb{G}(1, n+1)$, with **J. C. Sierra** and **L. Ugaglia** (2005).

Giornate di Geometria Algebrica e argomenti correlati V, Gargano, Italy, 2000

Projections of Grassmannians

- **Classification of n-dimensional subvarieties of $\mathbb{G}(1,2n)$ that can be projected to** $\mathbb{G}(1,n+1)$, with **Jose Carlos Sierra** and **Luca Ugaglia** (2005).
  - Classification of varieties that are projectable to $\mathbb{P}^{2n}$ coming from $\mathbb{P}^{n(n+3)/2-1}$.
  - Classification of varieties that are projectable to $\mathbb{G}(1,n+1)$ coming from $\mathbb{G}(1,2n)$.
  - The Veronese varieties and their inner projections are the only projectable varieties from $\mathbb{G}(1,2n)$ to $\mathbb{G}(1,n+1)$, under certain conditions.
  - For the projective case?
- **Characterization of Veronese varieties via projections in Grassmannians**, with *Raffaella Paoletti* (2005).
  - Extension to $\mathbb{G}(d-1,nd+d-1)$ into $\mathbb{G}(d-1,n+2d-3)$.

Steiner and Schwarzenberger bundles

- **Schwarzenberger bundles of arbitrary rank on the projective space**, on his own (2010).
  - Introduction of a certain class of Steiner bundles generalizing the construction of Schwarzenberger (generalized Schwarzenberger bundle).
  - When is a Steiner bundle a generalized Schwarzenberger bundle?
  - Notion of jumping subspaces of a Steiner bundle.
  - Characterization of the generalized Schwarzenberger bundles as the Steiner bundles whose set of jumping hyperplanes have maximal dimension.

Steiner and Schwarzenberger bundles

- **Jumping pairs of Steiner bundles**, with **Simone Marchesi** (2015).

Simone's wedding, Varese, Agoust 1st, 2011

Steiner and Schwarzenberger bundles

- **Jumping pairs of Steiner bundles**, with **Simone Marchesi** (2015).
  - General definition of a Steiner bundle on a Grassmannian finding lower bounds for its possible ranks.
  - Notion of a Schwarzenberger bundle on a Grassmannian.
  - *Jean Vallès* found a characterization of Steiner bundles which are also Schwarzenberger bundles in terms of jumping hyperplanes.
  - Notion of jumping pairs associated to a Steiner bundle which generalizes the concept of jumping hyperplanes.
  - They classify all Steiner bundles on $\mathbb{G}(k, n)$ whose jumping locus is maximal and show that they are also Schwarzenberger bundles.
- **Schwarzenberger bundles on smooth projective varieties**, with *Simone Marchesi* and *Helena Soares* (2016).

More algebraic geometry topics

- **On the variety parameterizing completely decomposable polynomials**, with *Alessandra Bernardi* (2011).
  - Variety of splitting forms, that is, the variety whose points are classes of degree d forms splitting as a product of d linear forms in $n+1$ variables $\longleftrightarrow \mathbb{G}(n-1, n+d-1)$.
  - Results concerning the higher secant varieties of the varieties of splitting forms.
- **Skew-symmetric tensor decomposition**, with *Alessandra Bernardi*, *Pedro Macias Marques* and *Bernard Murrain* (2021)
  - From the algebraic geometry point of view, studying skew-symmetric tensor decomposition is related to the study of higher secant varieties of Grassmannians.
  - From the skew-symmetric action $\rightsquigarrow$ skew-catalecticant matrices $\rightsquigarrow$ skew apolarity lemma (analogue of the classical apolarity lemma for symmetric tensors).

More algebraic geometry topics

- **A notion of Delta-multigenus for certain rank two ample vector bundles**, with *Antonio Lanteri* and *Carla Novelli* (2013).

Europroj meeting "EuroConference, Vestkapp, Norway, 1995

Following Hartshorne's conjecture

- *Hartshorne's conjecture*: A smooth n-dimensional subvariety $X \subset \mathbb{P}^N$ must be a complete intersection if $2N < 3n$.
  - Any rank-2 vector bundle over a projective space $\mathbb{P}^n$ splits into the direct sum of two line bundles.
- *W. Barth* and *M.E. Larsen* (1970, 1972, 1973): "The cohomology of X behaves like the one of a complete intersection".
- It implies that the Picard group of such X is generated by the class of its hyperplane section if $N \leq 2n - 2$.
- One can replace the projective space with other ambient spaces.
- The main evidence for this conjecture is cohomological.

Following Hartshorne's conjecture

- **Subcanonicity of codimension two subvarieties** on his own (2005).
  - Conjecture in the codimension two: for $N \geq 6$, any codimension two smooth $X \subset \mathbb{P}^N$ has to be a complete intersection.
  - Barth-Larsen theorem $\rightsquigarrow$ any codimension 2 variety in these conditions is subcanonical.
  - Study of the subcanonicity problem of codimension two subvarieties contained in Grassmannians of lines or quadrics (as ambient spaces).
- **Evidence to subcanonicity of codimension two submanifolds of** $\mathbb{G}(1,4)$, with *Maria Lucia Fania* (2006).
  - Any smooth codimension two projective subvariety of $\mathbb{G}(1,4)$ of degree ≤ 25 is subcanonical (classification of such subvarieties).
  - Consequence, any smooth codimension two projective subvariety of $\mathbb{G}(1,4)$ which is not of general type has degree ≤ 32 (classification of such subvarieties).

Following Hartshorne's conjecture

- **On the Picard group of low-codimension subvarieties**, with **Jorge Caravantes** (2009).

PhD defense, Universidad Complutense de Madrid, January, 2006

  - How to determine if n-dimensional smooth subvarieties of an ambient space of dimension at most $2n-2$ inherit the Picard group from the ambient space? Key step: the subvariety is simply connected?
  - Ambient space is a $\mathbb{G}(1, n)$ or a product of two projective spaces of the same dimension.

How Enrique understands algebraic geometry

- **Deriving the Serre correspondence by hand** (2004).
- **Another elementary proof of the Nullstellensatz** (2006).
- **A home-made Hartshorne-Serre correspondence** (2007).
- **Representation theory of finite groups through (basic) algebraic geometry** (2023).

One more student

- PhD title: **Matrices de Gram y espacios de ángulos diédricos de poliedros**, Raquel Díaz Sánchez, july 5th 1996.

PhD defense, Universidad Complutense de Madrid, July, 1996

One year earlier

Casa Galicia, Paris, spring, 1995

Latin Quartier, Paris, spring, 1995

Summer schools

- Perugia 1998, 2001, 2007, 2011, 2016.

Summer School SMI, Perugia, 2011

Summer schools

- Perugia 1998, 2001, 2007, 2011, 2016.

Summer School SMI, Perugia, 2016

Young mathematician spirit forever

Homemade Algebraic Geometry, Celebrating Enrique Arrondo's 60th birthday, Alcalá de Henares, July 11, 2023

Homemade algebraic geometry

Homemade Algebraic Geometry, Celebrating Enrique Arrondo's 60th birthday, Alcalá de Henares, July 12, 2023

Homemade algebraic geometry

But this is not over yet!!!