Every finite group is the group of automorphism of an evolution algebra

Cristina Costoya, Panagiote Ligouras, **Alicia Tocino*** and Antonio Viruel

*Departamento de Matemática Aplicada, Universidad de Málaga, Spain

*The author is supported by the Spanish Ministerio de Ciencia e Innovación through the project PID2019-104236GB-I00 and by the Junta de Andalucía through the projects FQM-336 and UMA18-FEDERJA-110, all of them with FEDER funds.

Abstract

We show that evolution algebras over any given field $\Bbbk$ are finitely universal. In other words, given any finite group G, there exist infinitely many regular and simple evolution algebras X such that $\mathrm{Aut}(X) \cong G$. We also show that any constant finite algebraic affine group scheme $\mathbf{G}$ over $\Bbbk$ is isomorphic to the algebraic affine group scheme of automorphisms of a regular evolution algebra.

Motivation

Finitely universal

Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. $\mathrm{Aut}_{\mathcal{C}}(X)$ is a group. If every finite group G can be represented as $\mathrm{Aut}_{\mathcal{C}}(X) \cong G$ for some $X \in \mathrm{Obj}(\mathcal{C})$ we say that $\mathcal{C}$ is *finitely universal* (f.u.). Identifying f.u. categories is a hard problem:

- $\mathcal{G}roups$ is not f.u.
- $\mathcal{G}raphs$ (finite simple non oriented) is f.u.
- $\mathcal{A}lg_{\Bbbk}$ (unital commutative associative $\Bbbk$-algebras) is f.u.

Evolution algebras (e.a.)

- E.a. model self-reproduction in non-Mendelian genetics.
- Example of Mendelian genetics: E and e two alleles with multiplication table of the gametic algebra given by

	E	e
E	E	$\frac{1}{2}$(E + e)
e	$\frac{1}{2}$(E + e)	e

- Example of non-Mendelian genetics: a_1 and a_2 two alleles with multiplication table of the gametic algebra given by

	a_1	a_2
a_1	$\omega_{11}a_1 + \omega_{21}a_2$	0
a_2	0	$\omega_{12}a_1 + \omega_{22}a_2$

with $\omega_{11} + \omega_{21} = \omega_{12} + \omega_{22} = 1$.

Basics on evolutions algebra

- An *evolution algebra* over a field $\Bbbk$ is a $\Bbbk$-algebra X provided with a basis $B = \{b_i \,|\, i \in \Lambda\}$ such that $b_i b_j = 0$ whenever $i \neq j$. Such a basis B is called a *natural basis*.
- Let X be an e.a. with natural basis B. The scalars $\omega_{ki} \in \Bbbk$ such that
$$b_i^2 := b_i b_i = \Sigma_{k\in\Lambda}\omega_{ki}b_k$$
are called the *structure constants* of X relative to B, and the matrix $M_B := (\omega_{ki})$ is said to be the *structure matrix* of X relative to B.
- For more definitions and properties see [1].

Evolution algebras and graphs

- Let X be a finite dimensional e.a. over $\Bbbk$ with natural basis B and structure matrix M_B. The directed graph $\Gamma(X, B) = (V, E)$ with $V = \{1, \ldots, n\}$, $E = \{(i, j) \in V \times V : \omega_{ij} \neq 0\}$ is called the oriented graph attached to the e.a. X relative to B.
- Given a simple graph $\mathcal{G} = (V, E)$, we define $\mathcal{X}(\mathcal{G})$ to be the e.a. over $\Bbbk$, with natural basis
$$B = \{b_v : v \in V\} \cup \{b_e : e \in E\}$$
and multiplication given by $b_v^2 = b_v$ for all $v \in V$ and $b_e^2 = b_e + \Sigma_{v\in e}b_v$ for all $e \in E$.

Main result

Let $\mathcal{E}_{\Bbbk}$ be the category of evolution algebras over a field $\Bbbk$ and G be a finite group. Then there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_{\Bbbk})$ such that X is regular or simple (if $\Bbbk$ is large enough), and $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X) \cong G$. In particular the category $\mathcal{E}_{\Bbbk}$ is finitely universal.

For regular evolution algebras

An e.a. X is *regular* if and only if $X = X^2$, or equivalently, if and only if for any natural basis B, the structure matrix M_B is a regular matrix.

- Given any field $\Bbbk$, there exists a covariant faithful functor $\mathcal{X} \colon \mathcal{SG}raphs \rightsquigarrow \mathcal{E}_{\Bbbk}$ such that for every $\mathcal{G} \in \mathcal{SG}raphs$ the following hold: $\mathcal{X}(\mathcal{G})$ is regular, and $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathcal{X}(\mathcal{G})) \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$.
- For any simple graph $\mathcal{G}$, the e.a. $\mathcal{X}(\mathcal{G})$ is regular.
- Given a simple graph $\mathcal{G} = (V, E)$, and $g \in \mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathcal{X}(\mathcal{G}))$, there exists $f \in \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ such that $g = \mathcal{X}(f)$.
- Given G a finite group, then there exists an e.a. X such that X is regular and $\mathrm{Aut}(X) \cong G$.
- For more details scan the QR code in the references.

For simple evolution algebras

An e.a. X is *simple* if and only if $X^2 \neq 0$ and 0 is the only proper ideal.

- $G \leq \mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathrm{Cay}(f))$.
- If $\mathrm{Cay}(f)$ is regular and $S = \mathrm{supp}(f)$ generates G, then $\mathrm{Cay}(f)$ is simple.
- Given G a finite group, if $\mathrm{Cay}(f)$ is regular, $S = \mathrm{supp}(f)$ generates G, $f|_S$ injective and in S there are elements of coprime order, then $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(\mathrm{Cay}(f)) \cong G$.
- If $\Bbbk$ is a large enough field, given a finite group G and any $S \subset G$, there exists a set-theoretical map $f : G \to \Bbbk$ such that $S = \mathrm{supp}(f)$, $f|_S$ is injective and $\mathrm{Cay}(f)$ is regular.
- Given G a finite group, if $\Bbbk$ is large enough, then there exists an e.a. X such that X is simple and $\mathrm{Aut}(X) \cong G$.
- More details soon.

Cayley evolution algebras

Given a finite group G and a field $\Bbbk$ one can consider a set-theoretical map $f : G \to \Bbbk$. The *Cayley e.a. associated to* f is the algebra $\mathrm{Cay}(f)$ given by the $\Bbbk-$vectorial space, $\Bbbk[G]$, provided with a natural basis $B = \{g \,|\, g \in G\}$ such that $g \cdot h = 0$ whenever $g \neq h$ and $g \cdot g = \Sigma_{h\in G}f(h)gh$.

Finite algebraic affine group scheme

Any e.a. X over $\Bbbk$ defines an algebraic affine group scheme over $\Bbbk$, $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X)\colon \mathcal{A}lg_{\Bbbk} \rightsquigarrow \mathcal{G}roups$, that takes any $R \in \mathrm{Obj}(\mathcal{A}lg_{\Bbbk})$ to $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X_R)$, the group of R-algebra automorphisms of $X_R = X \otimes_{\Bbbk} R$.

Theorem. Given any field $\Bbbk$, and any constant finite algebraic affine group scheme $\mathbf{G}$ over $\Bbbk$, there are infinitely many non isomorphic $X \in \mathrm{Obj}(\mathcal{E}_{\Bbbk})$ such that the group scheme $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X)$ is isomorphic to $\mathbf{G}$.

Main idea. Let $G = \mathbf{G}(\Bbbk)$. There exists a graph $\mathcal{G}$ such that $\mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong G$. Consider the e.a. $X = \mathcal{X}(\mathcal{G}) \in \mathrm{Obj}(\mathcal{E}_{\Bbbk})$. There exists an exact sequence of algebraic affine group schemes (see [2])
$$1 \to \mathbf{Diag}(\Gamma) \to \mathbf{Aut}_{\mathcal{E}_{\Bbbk}}(X) \to \mathrm{Aut}(\Gamma)$$
where $\Gamma = \Gamma(X, B)$ is the oriented graph attached to X, $\mathbf{Diag}(\Gamma)$ is the diagonal group scheme of Γ that turns out to be 1, $\mathrm{Aut}(\Gamma)$ is the constant group scheme given by the group of oriented graph automorphisms of Γ and $\mathrm{Aut}_{\mathcal{E}_{\Bbbk}}(X) \cong \mathrm{Aut}_{\mathcal{G}raphs}(\mathcal{G}) \cong \mathbf{G}$.

References

[1] Yolanda Cabrera Casado, Mercedes Siles Molina, and M. Victoria Velasco, *Evolution algebras of arbitrary dimension and their decompositions*, Linear Algebra Appl. 495 (2016), 122-162.

[2] Alberto Elduque and Alicia Labra, *Evolution algebras, automorphisms, and graphs*, Linear and Multilinear Algebra, Vol. 69, Núm. 2, pp. 331-342 (2021).