On the vanishing of the hyperdeterminant under certain symmetry conditions

Joint work with Enrique Arrondo

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⊛ Summary

V vector space over a field $\mathbb{K}$ with characteristic coprime with $d!$

$$V^* \otimes^{(d)} \cdots \otimes V^* = \{ F : V \times^{(d)} \cdots \times V \longrightarrow \mathbb{K} \text{ multilinear form} \}$$

First decompose $V^* \otimes^{(d)} \cdots \otimes V^*$ into pieces (symmetry classes) W_λ according to the different partitions λ of d.

$$\Downarrow$$

$$F = F_1 + F_2 + \cdots + F_k \,, \quad F_i \in W_\lambda$$

Second compute the hyperdeterminant of multilinear forms in each symmetry class.

⊛ Symmetry classes

Definition: $W \leq V^* \otimes \overset{(d)}{\cdots} \otimes V^*$ is a symmetry class if $\sigma F \in W$, $\forall F \in W$, $\forall \sigma \in S_d$.

$$\sigma F(x_1, \dots, x_d) = F(x_{\sigma(1)}, \dots, x_{\sigma(d)}), \quad \sigma \in S_d.$$

Example: $d = 2$

$$F(x_1, x_2) = \underbrace{\frac{F(x_1,x_2) + F(x_2,x_1)}{2}}_{=: F_S(x_1,x_2)} + \underbrace{\frac{F(x_1,x_2) - F(x_2,x_1)}{2}}_{=: F_A(x_1,x_2)}$$

F_S: symmetric

F_A: skew-symmetric

$$F = F_S + F_A$$

$$\begin{cases} F \in V^* \otimes V^* \\ F_S \in W_{(2)}, \; F_A \in W_{(1,1)}. \end{cases}$$

Partitions ⟷ Young diagrams

(2)

(1,1)

$$V^* \otimes V^* = W_{(2)} \oplus W_{(1,1)} \rightsquigarrow \begin{cases} W_{(2)} = \{F : V \times V \to \mathbb{K} \;/\; F = \sigma F, \; \forall \sigma \in S_2\} \\ W_{(1,1)} = \{F : V \times V \to \mathbb{K} \;/\; F = \mathrm{sg}(\sigma)\, \sigma F, \; \forall \sigma \in S_2\} \end{cases}$$

Metropolis - Rota - Stein, 1991

equations for symmetry classes for $d = 3$ and in general using Young diagrams

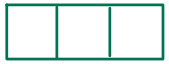
Example: d=3

$$F = \underbrace{\frac{1}{6}\sum_{\sigma\in S_3}\sigma F}_{=F_S} + \underbrace{\frac{1}{6}\sum \mathrm{sg}(\sigma)\sigma F}_{=F_A} + \underbrace{F - \frac{1}{3}\left(F+(123)F+(132)F\right)}_{=F_Y}$$

F_S: symmetric

F_A: skew-symmetric

F_Y: cyclic or Y-symmetric



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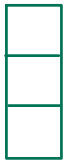
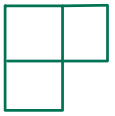


(1,1,1)



(2,1)

STANDARD REPRESENTATION (d-1,1)

$$V^*\otimes V^*\otimes V^* = W_{(3)}\oplus W_{(1,1,1)}\oplus W_{(2,1)} \rightsquigarrow \begin{cases} W_{(3)} = \{F: V\times V\to \mathbb{K} \;/\; F=\sigma F,\ \forall \sigma\in S_3\} \\ W_{(1,1,1)} = \{F: V\times V\to \mathbb{K} \;/\; F=\mathrm{sg}(\sigma)\sigma F,\ \forall \sigma\in S_3\} \\ W_{(2,1)} = \{F: V\times V\to \mathbb{K} \;/\; F+(123)F+(132)F=0\} \end{cases}$$

Metropolis - Rota - Stein, 1991

→ For a general d :

① Fulton-Harris (Representation Theory: a First Course, 1991) ⇝ Schur Functors

$$V^* \otimes^{(d)} \cdots \otimes V^* \cong \bigoplus_{\lambda} \left(S_\lambda V\right)^{\otimes m_\lambda}$$

$m_\lambda = \dim V_\lambda$ irreducible representation of S_d corresponding to λ.

② Enrique Arrondo (Representation Theory of Finite Groups through (basic) Algebraic Geometry, 2021).

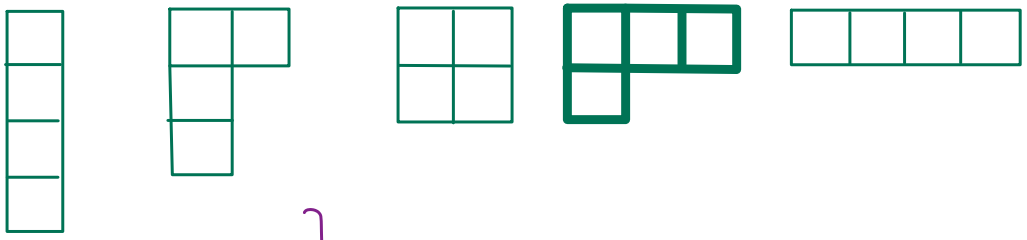
$$V^* \otimes^{(d)} \cdots \otimes V^* \cong \bigoplus_{\lambda} W_\lambda$$

sketch of this method

Example: $d = 4$

→ conjugacy classes of S_4 ⇝ $\overline{(1)}$ $\overline{(12)}$ $\overline{(12)(34)}$ $\overline{(123)}$ $\overline{(1234)}$

partitions and Young diagrams $(1,1,1,1)$ $(2,1,1)$ $(2,2)$ $(3,1)$ (4)



→ Group algebra $\mathbb{K}(S_4) = \left\{ \sum a_\sigma e_\sigma : a_\sigma \in \mathbb{K} \right\}$ with $e_\sigma e_\tau = e_{\sigma\tau}$.

→ Basis of $\mathbb{K}(S_4)$:

- $\alpha_{(1,1,1,1)} = e_{(1)}$
- $\alpha_{(2,1,1)} = e_{(12)} + e_{(13)} + e_{(14)} + e_{(23)} + e_{(24)} + e_{(34)}$
- $\alpha_{(2,2)} = e_{(12)(34)} + e_{(13)(24)} + e_{(14)(23)}$
- $\alpha_{(3,1)} = e_{(123)} + e_{(132)} + e_{(124)} + e_{(142)} + e_{(134)} + e_{(143)} + e_{(234)} + e_{(243)}$
- $\alpha_{(4)} = e_{(1234)} + e_{(1243)} + e_{(1324)} + e_{(1342)} + e_{(1423)} + e_{(1432)}$

→ Basis of $\mathbb{Z}K(S_4)$:

- $\alpha_{(1,1,1,1)} = c_{(1)} =: \mathbf{1}$
- $\alpha_{(2,1,1)} = c_{(12)} + c_{(13)} + c_{(14)} + c_{(23)} + c_{(24)} + c_{(34)} =: \mathbf{x}$
- $\alpha_{(2,2)} = c_{(12)(34)} + c_{(13)(24)} + c_{(14)(23)} =: \mathbf{w}$
- $\alpha_{(3,1)} = c_{(123)} + c_{(132)} + c_{(124)} + c_{(142)} + c_{(134)} + c_{(143)} + c_{(234)} + c_{(243)} =: \mathbf{y}$
- $\alpha_{(4)} = c_{(1234)} + c_{(1243)} + c_{(1324)} + c_{(1342)} + c_{(1423)} + c_{(1432)} =: \mathbf{z}$

→ Multiplication in $\mathbb{Z}K(S_4)$ as K-algebra:

$x^2 = 6 + 3y + 2w$ $\quad$ $yz = 4x + 4z$ $\quad$ $zw = 2x + z$

$xy = 4x + 4z$ $\quad$ $z^2 = 6 + 3y + 2w$ $\quad$ $w^2 = 3 + 2w$

$y^2 = 8 + 4y + 8w$ $\quad$ $xw = x + 2z$

$xz = 3y + 4w$ $\quad$ $yw = 3y$

→ $I = \text{span}(x^2-6-3y-2w,\ xy-4x-4z,\ y^2-8-4y-8w,\ xz-3y-4w,$

$yz-4x-4z,\ z^2-6-3y-2w,\ xw-x-2z,\ yw-3y,\ zw-2x-z,\ w^2-3-2w)$

→ $V(I) = \{(6,8,6,3), (-6,8,-6,3), (0,-4,0,3), (-2,0,2,-1), (2,0,-2,-1)\}$.

→ There exists an isomorphism between $\mathbb{Z}\mathbb{K}(S_4)$ and $\mathbb{K}^5$:

$$\mathbb{Z}\mathbb{K}(S_4) \longrightarrow \mathbb{K}[x,y,z,w]/I \longrightarrow \mathbb{K}^5$$

1	⟼	$\bar{1}$	⟼	$(1,1,1,1,1)$
$\alpha_{(2,1,1)}$	⟼	$\bar{x}$	⟼	$(6,-6,0,-2,2)$
$\alpha_{(3,1)}$	⟼	$\bar{y}$	⟼	$(8,8,-4,0,0)$
$\alpha_{(4)}$	⟼	$\bar{z}$	⟼	$(6,-6,0,2,-2)$
$\alpha_{(2,2)}$	⟼	$\bar{w}$	⟼	$(3,3,3,-1,-1)$

primitive idempotents ⟷ canonical basis

→ There exists an isomorphism between $\mathbb{Z}K(S_4)$ and K^5:

$$\mathbb{Z}K(S_4) \longrightarrow K^5$$

$$\beta_S = \frac{1}{24}\left(1+\alpha_{(2,1,1)}+\alpha_{(3,1)}+\alpha_{(4)}+\alpha_{(2,2)}\right) \longrightarrow (1,0,0,0,0)$$

$$\beta_A = \frac{1}{24}\left(1-\alpha_{(2,1,1)}+\alpha_{(3,1)}-\alpha_{(4)}+\alpha_{(2,2)}\right) \longrightarrow (0,1,0,0,0)$$

$$\beta_1 = \frac{1}{24}\left(4-2\alpha_{(3,1)}+4\alpha_{(2,2)}\right) \longrightarrow (0,0,1,0,0)$$

$$\beta_2 = \frac{1}{24}\left(9+3\alpha_{(2,1,1)}+3\alpha_{(4)}-3\alpha_{(2,2)}\right) \longrightarrow (0,0,0,1,0)$$

$$\beta_Y = \frac{1}{24}\left(9+3\alpha_{(2,1,1)}-3\alpha_{(4)}-3\alpha_{(2,2)}\right) \longrightarrow (0,0,0,0,1)$$

primitive idempotents ⟷ canonical basis

$\beta_S, \beta_A, \beta_1, \beta_2, \boxed{\beta_Y}$ → standar representation

→ $$F = \beta_S F + \beta_A F + \beta_1 F + \beta_2 F + \beta_Y F$$

→ $$V^* \otimes V^* \otimes V^* \otimes V^* = W_S \oplus W_A \oplus W_1 \oplus W_2 \oplus W_Y$$ ⇝

$$\begin{cases} W_S = \beta_S K(S_4) \\ W_A = \beta_A K(S_4) \\ W_1 = \beta_1 K(S_4) \\ W_2 = \beta_2 K(S_4) \\ W_Y = \beta_Y K(S_4) \end{cases}$$

all these equations are the same as those given by Metropolis-Rota-Stein obtained in a completely different way.

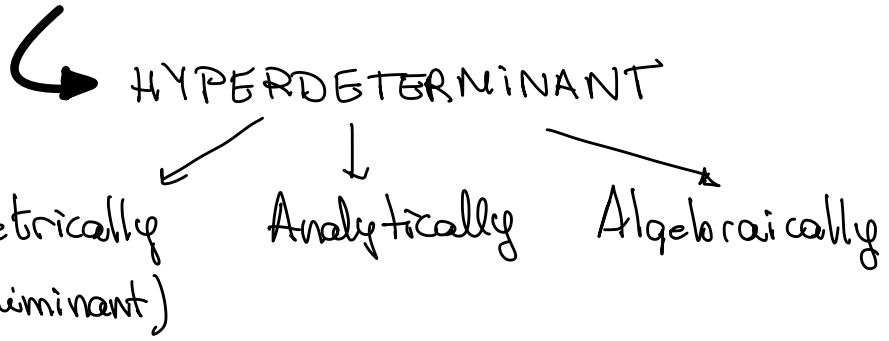
⊛ Hyperdeterminant

→ Cayley ⇝ determinants 1843 ⇝ generalization 1845 ⇝

⇝ On the theory of linear transformations.

→ Gelfand, Kapranov and Zelevinski 1994 ⇝

⇝ Discriminants, Resultants and Multidimensional determinants.



Notion of hyperdeterminant of a general hypermatrix

⇕

Notion of hyperdeterminant of a multilinear form

$F : V_1 \times \cdots \times V_d \longrightarrow \mathbb{K}$

→ Ottaviani 2012 ⇝ Introduction to the hyperdeterminant and to the rank of multidimensional matrices.

Definition: $\dim V = n$, $\mathbb{P}^{n-1} \times \overset{(d)}{\cdots} \times \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}(V^* \otimes \overset{(d)}{\cdots} \otimes V^*)$ Segre embedding

The hyperdeterminant of format $n \times \overset{(d)}{\cdots} \times n$ is a homogeneous polynomial function on $V^* \otimes \overset{(d)}{\cdots} \otimes V^*$ which is the defining equation of the projectively dual variety $\left(\mathbb{P}^{n-1} \times \overset{(d)}{\cdots} \times \mathbb{P}^{n-1}\right)^\vee$.

That is, $\mathrm{Det} : V^* \otimes \cdots \otimes V^* \longrightarrow \mathbb{K}$

$$F \longmapsto \mathrm{Det}(F)$$

Question: How do we compute it? Not easy :(

- explicit formulas only in some cases (boundary format case)
- elimination theory needed
- geometric interpretation

Example of computation: $2\times2\times2$ (GKZ)

$$V^* \overset{(d)}{\otimes \cdots \otimes} V^* = \{F : V \times \overset{(d)}{\cdots} \times V \longrightarrow \mathbb{K} \text{ multilinear form}\}$$

Fix $B = \{v_1, \dots, v_n\}$ basis of V and define

$$a_{i_1 \dots i_d} = F(v_{i_1}, \dots, v_{i_d})$$

$$A = (a_{i_1 \dots i_d}) \Longleftrightarrow F : V \overset{(d)}{\times \cdots \times} V \longrightarrow \mathbb{K}$$

The $2\times2\times2$ hyperdeterminant appears in physics literature.

For example in:

Duff, M.J., String triality, black hole entropy, and Cayley's hyperdeterminant, 2007.

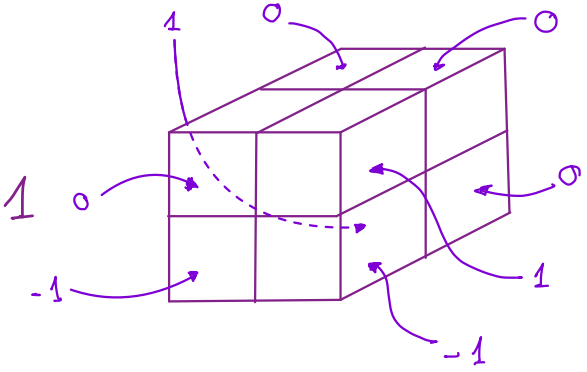
Proposition 1.7. *The hyperdeterminant of a matrix* $A = (a_{ijk})$ $(i, j, k = 0, 1)$ *is given by*

$$\begin{aligned}\mathrm{Det}(A) = {} & (a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2) \\ & -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} \\ & +a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) \\ & +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).\end{aligned} \tag{1.5}$$

$F: \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K}^2 \longrightarrow \mathbb{K}$ multilinear form given by

$$F((x_1,y_1),(x_2,y_2),(x_3,y_3)) = x_1x_2y_3 - x_1y_2x_3 + y_1y_2x_3 - x_1y_2y_3$$

↳ multidimensional matrix $A = (a_{ijk})$, $i,j,k = 0,1$



$a_{000} = 0$ $\quad a_{001} = 1$ $\quad a_{100} = 0$ $\quad a_{101} = 0$

$a_{010} = -1$ $\quad a_{011} = -1$ $\quad a_{110} = 1$ $\quad a_{111} = 0$

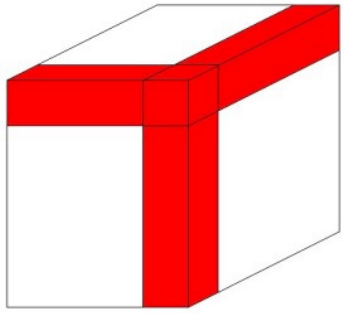
first slice / second slice

$$Det(F) = \left(\begin{vmatrix} a_{000} & a_{011} \\ a_{100} & a_{111} \end{vmatrix} + \begin{vmatrix} a_{010} & a_{001} \\ a_{110} & a_{101} \end{vmatrix} \right)^2 - 4 \begin{vmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{vmatrix} \cdot \begin{vmatrix} a_{010} & a_{011} \\ a_{110} & a_{111} \end{vmatrix} =$$

Example 6 Ottaviani

$$= \left(\begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \right) - 4 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

Example of geometric interpretation: $n \times n \times n$ (0)



A matrix A is degenerate if and only if it becomes zero on the colored part after a linear change of coordinates.

We want to know **when $\mathrm{Det}(F)=0$**

Property: F is degenerate $\iff \mathrm{Det}(F)=0 \iff$

$\iff$ there exist $v_1,\dots,v_d \in V$ such that

$$F(v_1,\dots,v_{d-1},V)=F(v_1,\dots,V,v_d)=\cdots=F(V,v_2,\dots,v_d)=0.$$

In particular, $\mathrm{Det}(F)=0 \iff$ there exists $v\in V$ such that

$$F(v,\dots,v,V)=F(v,\dots,V,v)=\cdots=F(V,v,\dots,v)=0.$$

With the previous property and using the equations to describe the symmetry classes,

$$V^*\otimes^{(d)}\cdots\otimes V^*=\bigoplus_\lambda W_\lambda \Rightarrow F=F_{\lambda_1}+F_{\lambda_2}+\cdots+F_{\lambda_K} \text{ with } F_{\lambda_k}\in W_{\lambda_k}.$$

we have the result

RESULT:

If $F \in W_\lambda$ with $\lambda \neq (d), (d-1,1)$, then $\mathrm{Det}(F) = 0$.

Moreover, if $F \in \bigoplus_{\lambda \neq (d),(d-1,1)} W_\lambda$ then $\mathrm{Det}(F) = 0$.

→ 2016 ⇝ 3 months with Ottaviani and using Metropolis-Rota-Stein.

→ 2021 ⇝ Arrondo with his method to describe symmetry classes.

No information for $d = 2$:

$W_{(2)}$ ⇝ symmetric matrices

$W_{(1,1)}$ ⇝ skew-symmetric matrices.

Example:

$d = 3$ $$F = \underbrace{\frac{1}{6}\sum_{\sigma \in S_3} \sigma F}_{W_S} + \underbrace{\frac{1}{6}\sum \mathrm{sg}(\sigma)\sigma F}_{W_A} + \underbrace{F - \frac{1}{3}\left(F + (123)F + (132)F\right)}_{W_Y}$$

Det = 0

W_S (3)

W_A (1,1,1)

W_Y (2,1)

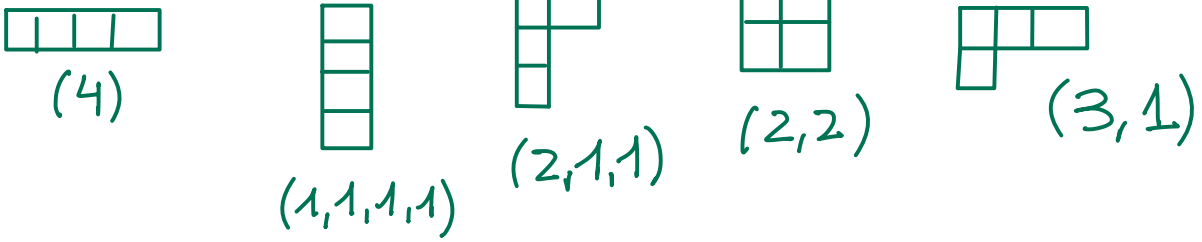
Example:

$d=4$

$$F = \beta_S F + \beta_A F + \beta_1 F + \beta_2 F + \beta_Y F$$

Det = 0

W_S W_A W_1 W_2 W_Y



BUT

It is known that the determinant of $n \times n$ skew-symmetric matrices vanishes when n is odd

$$\begin{vmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{vmatrix} = 0$$

So, inside $W_{(1,1)} = W_{(d-1,1)}$ there are some pieces where the (hyp)determinant should vanish.

Remark: Note that if $F \in W_{(d-1,1)}(V,\mathbb{K})$ then $\mathrm{Det}(F)$ is not necessarily zero

Example: $F: \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K}^2 \longrightarrow \mathbb{K}$ multilinear form given by

$$F((x_1,y_1),(x_2,y_2),(x_3,y_3)) = x_1x_2y_3 - x_1y_2x_3 + y_1y_2x_3 - x_1y_2y_3$$

We proved that $\mathrm{Det}(F) = -1 \neq 0$.

Observe that $F \in \underbrace{W_{(2,1)}(\mathbb{K}^2,\mathbb{K})}_{\text{''}}$ since

$$\{ F: \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K}^2 \longrightarrow \mathbb{K}\ ,\ F + (123)F + (132)F = 0 \}$$

$$F((x_1,y_1),(x_2,y_2),(x_3,y_3)) + F((x_2,y_2),(x_3,y_3),(x_1,y_1)) + F((x_3,y_3),(x_1,y_1),(x_2,y_2)) =$$

$$x_1x_2y_3 - x_1y_2x_3 + y_1y_2x_3 - x_1y_2y_3 + x_2x_3y_1 - x_2y_3x_1 + y_2y_3x_1 - x_2y_3y_1 +$$

$$+ x_3x_1y_2 - x_3y_1x_2 + y_3y_1x_2 - x_3y_1y_2 = 0$$

Improvement (2024): (Arrondo, T.)

(1) $W_{(d-1,1)} = \bigoplus_{i=1}^{d-1} W^i_{(d-1,1)}$ with $W^i_{(d-1,1)} = \{F \in W_{(d-1,1)} \;/\; \sigma F = \omega^i F\}$

where σ is a d-cycle and ω is primitive d-root of unity.

$F = H_1 + \cdots + H_{d-1}$ with $H_i \in W^i_{(d-1,1)}$

Sketch of the proof:

- $F \in W_{(d-1,1)} \Rightarrow F + \sigma F + \sigma^2 F + \cdots + \sigma^{d-1} F = 0$
- $\sum_{k=1}^{d-1} (\omega^k)^i = -1 \quad \forall\, i \in \{1, \dots, d-1\}$
- decomposition for $d = 4$.

$$F = \underbrace{\tfrac{1}{4}(F + \omega\sigma F + \omega^2\sigma^2 F + \omega^3\sigma^3 F)}_{H_3} + \underbrace{\tfrac{1}{4}(F + \omega^2\sigma F + (\omega^2)^2\sigma^2 F + (\omega^2)^3\sigma^3 F)}_{H_2} +$$

($\omega^4 = 1$, $\omega^6 = \omega^2$)

$$+ \underbrace{\tfrac{1}{4}(F + \omega^3\sigma F + (\omega^3)^2\sigma^2 F + (\omega^3)^3\sigma^3 F)}_{H_1}$$

($\omega^6 = \omega^2$, $\omega^9 = \omega$)

⊛ $H_3 = \frac{1}{4}(F + \omega\sigma F + \omega^2\rho^2 F + \omega^3\sigma^3 F) \in W^3_{(3,1)} \Leftrightarrow \sigma H_3 = \omega^3 H_3$:

$\sigma H_3 = \frac{1}{4}(\sigma F + \omega\rho^2 F + \omega^2\rho^3 F + \omega^3 F)$

$\omega^3 H_3 = \frac{1}{4}(\omega^3 F + \sigma F + \omega\rho^2 F + \omega^2\rho^3 F)$))) ✓

$H_2 = \frac{1}{4}(F + \omega^2\sigma F + \sigma^2 F + \omega^2\rho^3 F) \in W^2_{(3,1)}(V,\mathbb{K}) \Leftrightarrow \sigma H_2 = \omega^2 H_2$:

$\sigma H_2 = \frac{1}{4}(\sigma F + \omega^2\rho^2 F + \sigma^3 F + \omega^2 F)$

$\omega^2 H_2 = \frac{1}{4}(\omega^2 F + \sigma F + \omega^2\rho^2 F + \sigma^3 F)$))) ✓

$H_1 = \frac{1}{4}(F + \omega^3\sigma F + \omega^2\sigma^2 F + \omega\sigma^3 F) \in W^1_{(3,1)}(V,\mathbb{K}) \Leftrightarrow \sigma H_1 = \omega H_1$:

$\sigma H_1 = \frac{1}{4}(\sigma F + \omega^3\rho^2 F + \omega^2\rho^3 F + \omega F)$

$\omega H_1 = \frac{1}{4}(\omega F + \sigma F + \omega^3\rho^2 F + \omega^2\rho^3 F)$))) ✓

⊛ We prove ⊕ by induction.

(2) If $F \in W^{k}_{(d-1,1)}$ with $k \in \{1,\dots,d-1\}$ then $\mathrm{Det}(F)=0$.

It does not depend on $\dim V = n$

Sketch of the proof:

$$F \in W^{k}_{(d-1,1)} \Rightarrow \begin{cases} \sigma F = \omega^{k} F \\ F + \sigma F + \cdots + \sigma^{d-1} F = 0 \end{cases}$$

There exists $v \in V$ such that $F(v,\dots,v,V)=0$?

Since $\sigma F = \omega^{k} F \Rightarrow F(v,\dots,V,v) = \cdots = F(V,v,\dots,v) = 0 \rightsquigarrow \mathrm{Det}(F)=0$

Fix a basis of V, $B=\{v_1,\dots,v_n\}$ and take $v = \sum_{i=1}^{n} \lambda_i v_i$ $(\dim V = n)$

Since $F \notin W_{(d)} \Rightarrow F(x,\dots,x)=0$. Hence:

$$0 = F(v,\dots,v) = F\Big(\sum_{i=1}^{n}\lambda_i v_i, \dots, \underbrace{\sum_{i=1}^{n}\lambda_i v_i}_{\text{decomposing this summatory}}\Big) =$$

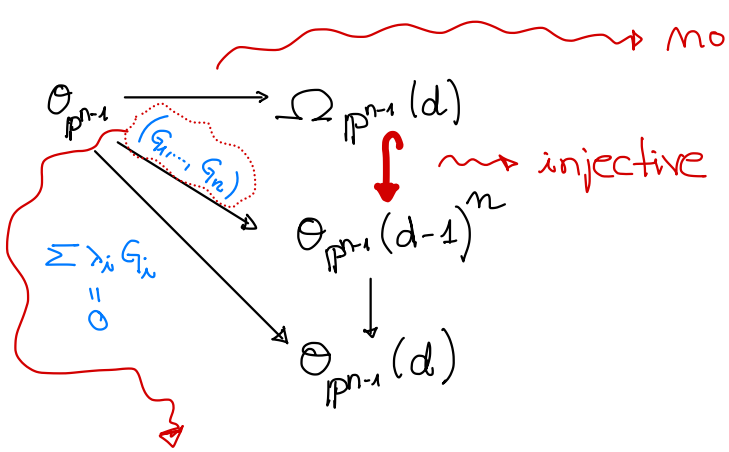
$$= \lambda_1 \underbrace{F\Big(\sum_{i=1}^{n}\lambda_i v_i, \dots, \sum_{i=1}^{n}\lambda_i v_i, v_1\Big)}_{G_1(\lambda_1,\dots,\lambda_n)} + \cdots + \lambda_n \underbrace{F\Big(\sum_{i=1}^{n}\lambda_i v_i, \dots, \sum_{i=1}^{n}\lambda_i v_i, v_n\Big)}_{G_n(\lambda_1,\dots,\lambda_n)}$$

$$0 = \lambda_1 \underbrace{F\Big(\sum_{i=1}^n \lambda_i v_i, \dots, \sum_{i=1}^n \lambda_i v_i, v_1\Big)}_{G_1(\lambda_1,\dots,\lambda_n)} + \dots + \lambda_n \underbrace{F\Big(\sum_{i=1}^n \lambda_i v_i, \dots, \sum_{i=1}^n \lambda_i v_i, v_n\Big)}_{G_n(\lambda_1,\dots,\lambda_n)} =$$

$$= \lambda_1 G_1(\lambda_1,\dots,\lambda_n) + \dots + \lambda_n G_n(\lambda_1,\dots,\lambda_n)$$

We look for non-trivial common solutions of the polynomials $G_1, \dots, G_n \in \mathbb{K}[\lambda_1, \dots, \lambda_n]_{d-1}$.

We interpret this as finding a point in $\mathbb{P}^{n-1}$ in the intersection of the corresponding n hypersurfaces.



NOT INJECTIVE $\Rightarrow$ there exist $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ such that

$G_1(\lambda_1,\dots,\lambda_n) = \dots = G_n(\lambda_1,\dots,\lambda_n) = 0$

we found $v = \sum \lambda_i v_i$ such that $F(v, \dots, v, V) = 0$ ☺

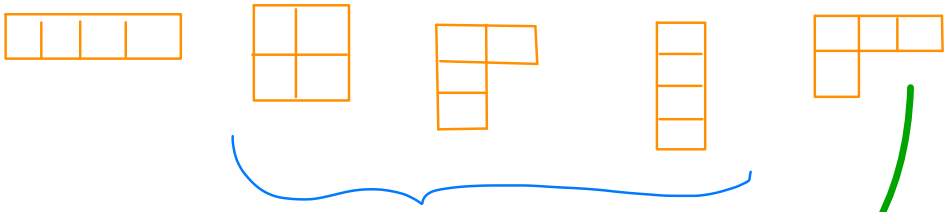
Main result:

V vector space over a field $\mathbb{K}$, fix $d \geq 3$.

Suppose $\operatorname{char}\mathbb{K}$ does not divide $d!$ and $\mathbb{K}$ contains a primitive d^{th}-root of unity, w.

If $F \in \left(\bigoplus_{\lambda \neq (d),(d-1,1)} W_\lambda \right) \oplus W_{(d-1,1)}^{\oplus i}$ for $i = 1, \dots, d-1$, then $\operatorname{Det}(F) = 0$.

$d=4$ $\quad V^* \otimes V^* \otimes V^* \otimes V^* = W_{(4)} \oplus W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}$



$\operatorname{Det}(F) = 0$

$W_{(3,1)} = W_{(3,1)}^1 \oplus W_{(3,1)}^2 \oplus W_{(3,1)}^3$

$\operatorname{Det}(F) = 0$

$F \in W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}^1$

$F \in W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}^2$

$F \in W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}^3$

$\Rightarrow \operatorname{Det} F = 0$

Thank you very much for your attention!