Tensor product of evolution algebras

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Contents of the talk

1. Preliminaries on evolution algebras
2. Generalities of the tensor product of algebras
3. Properties of the tensor product of evolution algebras
4. Perfect tensor product of evolution algebras

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Basics on evolution algebras

- J. P. Tian. **Evolution algebras and their applications**. Springer-Verlag Berlin Heidelberg, (2007)
- Y. Cabrera Casado. **Evolution algebras**. Doctoral dissertation. Universidad de Málaga (2016).
- Y. Cabrera Casado, M. Siles Molina and M. V. Velasco. **Evolution algebras of arbitrary dimension and their decompositions.** Linear Algebra Appl. 495 (2016).
- Y. Cabrera Casado, M. Kanuni and M. Siles Molina. **Basic ideals in evolution algebras**. Linear Algebra Appl. 570, (2019).

Definition

An **evolution algebra** over a field $\mathbb{k}$ is a $\mathbb{k}$-algebra A provided with a basis $B = \{e_i \mid i \in \Lambda\}$ such that $e_i e_j = 0$ whenever $i \neq j$. Such a basis B is called a **natural basis**.

Evolution algebras are **commutative** algebras but **not power associative** in general and **not associative** algebras.

Definition

Let A be an evolution algebra with natural basis B. The scalars $\omega_{ki} \in \mathbb{k}$ such that

$$e_i^2 := e_i e_i = \Sigma_{k \in \Lambda} \omega_{ki} e_k$$

are called the **structure constants** of A relative to B, and the matrix $M_B := (\omega_{ki})$ is said to be the **structure matrix** of A relative to B.

More definitions and properties

Relation between two structure matrices of A

$$M_{B'} = P^{-1} M_B P^{(2)}$$

where P is the change of basis matrix from B to B' and $P^{(2)} = (p_{ij}^2)$.

Definition

An algebra A is **simple** if $A^2 \neq 0$ and 0 is the only proper ideal.
An algebra A is **perfect** if $A^2 = A$. In the particular case of an evolution algebra, perfectness is equivalent to the fact that the determinant of the structure matrix is not zero.

A is evolution algebra:

perfect $\iff \det(M_B) \neq 0$

simple $\Rightarrow$ perfect

perfect $\not\Rightarrow$ simple

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Tensor product of vector spaces

Definition (Tensor product)

$$V \otimes W = \left\{ \sum_{i=1}^{n} \lambda_i v_i \otimes w_i \,|\, \lambda_i \in \mathbb{K},\ v_i \in V,\ w_i \in W,\ n \in \mathbb{N}^* \right\}$$

$$\begin{aligned} v \otimes w \ : \ V^* \times W^* &\to \Bbbk \\ (f, g) &\mapsto f(v)g(w) \end{aligned}$$

$$B_1 = \{e_i\}_{i\in\Lambda} \text{ basis of } V,\ B_2 = \{f_j\}_{j\in\Gamma} \text{ basis of } W \Longrightarrow$$
$$B_1 \otimes B_2 := \{e_i \otimes f_j\}_{i\in\Lambda, j\in\Gamma} \text{ basis of } V \otimes W$$

Definition

Let A_1 and A_2 be two $\mathbb{k}$-algebras with bases $\{e_i\}_{i\in\Lambda}$ and $\{f_j\}_{j\in\Gamma}$ respectively. We define a **product** on $A_1 \otimes A_2$ as follows:

$$(e_i \otimes f_j) \cdot (e_k \otimes f_r) = e_i e_k \otimes f_j f_r.$$

Simplicity and perfectness of tensor product of $\mathbb{k}$-algebras

Proposition

Let A_1 and A_2 be two $\mathbb{k}$-algebras and consider the tensor product $A_1 \otimes A_2$. Then

1. *A_1 and A_2 are perfect algebras $\iff$ $A_1 \otimes A_2$ is a perfect algebra.*
2. *$A_1 \otimes A_2$ is simple $\Rightarrow$ A_1 and A_2 are both simple algebras.*

Example

$\mathrm{char}(\mathbb{k}) \neq 2$, A is the 2-dimensional $\mathbb{k}$-algebra with basis $\{e_1, e_2\}$ such that $e_2e_1 = e_1e_2 = 0$, $e_1^2 = e_2$, $e_2^2 = e_1$. One can check that A is simple but $A \otimes A$ is not simple ($I = \mathrm{span}(\{e_1 \otimes e_1, e_2 \otimes e_2\})$ ideal of $A \otimes A$).

Definition

We say that a $\mathbb{k}$-algebra A is **tensorially decomposable** if it is isomorphic to $A_1 \otimes A_2$ where A_1 and A_2 are $\mathbb{k}$-algebras with $\dim(A_1), \dim(A_2) > 1$. Otherwise we say that A is **tensorially indecomposable**.

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Tensor product of evolution algebras

Question

Under which conditions being evolution algebra is inherited from the tensor product to the factors?

Theorem

Let A_1 and A_2 be $\mathbb{K}$-algebras.

1. *If A_1 and A_2 are evolution algebras, then $A_1 \otimes A_2$ is also an evolution $\Bbbk$-algebra. Furthermore, if $B_1 = \{e_i\}_{i\in\Lambda}$ and $B_2 = \{f_j\}_{j\in\Gamma}$ are natural bases of A_1 and A_2 respectively, then $\{e_i \otimes f_j\}_{i\in\Lambda, j\in\Gamma}$ is a natural basis of $A_1 \otimes A_2$.*
2. *Let A_1 and A_2 be commutative with $\mathrm{char}(\Bbbk) \neq 2$ and let $A_1 \otimes A_2$ be a finite dimensional evolution algebra. If A_1 and A_2 are* **locally nondegenerate**, *then A_1 and A_2 are evolution algebras.*

- Y. Cabrera Casado, C. Gil Canto, D. Martín Barquero and C. Martín González, **Simultaneous orthogonalization of inner products over arbitrary fields**, Springer Proc. Math. Stat., 427 (2023).

Structure matrix of tensor product of evolution algebras

$$A_1,\ A_2 \text{ e.a.} \rightsquigarrow B_1 = \{e_i\}_{i=1}^n,\ B_2 = \{f_j\}_{j=1}^m \rightsquigarrow M_{B_1} = (\omega_{ij}),\ M_{B_2} = (\beta_{kl})$$

$$A_1 \otimes A_2 \text{ e.a.} \rightsquigarrow B_1 \otimes B_2 = \{e_i \otimes f_j, \} \rightsquigarrow M_{B_1 \otimes B_2} = M_{B_1} \otimes M_{B_2}$$

Change of basis in tensor product

$B_1,\ B_1'$ natural bases of A_1

$B_2,\ B_2'$ natural bases of A_2

P is the change of basis matrix from B_1 to B_1'

Q is the change of basis matrix from B_2 to B_2'

$$\Downarrow$$

$P \otimes Q$ is a change of basis matrix from $B_1 \otimes B_2$ to $B_1' \otimes B_2'$

$$M_{B_1' \otimes B_2'} = (P \otimes Q)^{-1}(M_{B_1 \otimes B_2})(P \otimes Q)^{(2)}$$

Tensorially decomposable evolution algebras

Since **the decomposition of an algebra into tensor product is not unique**, it may happen that some evolution algebra A splits as $A_1 \otimes A_2$ with none of the A_i an evolution algebra and also $A = A_1' \otimes A_2'$ where both A_i' are evolution algebras.

Example

- $\mathbb{k}$ be a field with $\mathrm{char}\mathbb{k} \neq 2$.
- A, 2-dimensional $\mathbb{k}$-algebra, with basis $\{e_1, e_2\}$ such that $e_1^2 = e_2^2 = 0$ and $e_1 e_2 = e_2 = -e_2 e_1$. A is **not an evolution algebra**.
- $A \otimes A$ is an **evolution algebra** with natural basis $B = \{u_1 + u_4, u_1 - u_4, u_2 + u_3, u_2 - u_3\}$ (where $u_1 = e_1 \otimes e_1$, $u_2 = e_1 \otimes e_2$, $u_3 = e_2 \otimes e_1$ and $u_4 = e_2 \otimes e_2$).
- The structure matrix associated to $A \otimes A$ relative to B is

$$M_B = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 2 & -2 & -2 & 2 \end{array}\right) = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix}.$$

4-dimensional tensorially decomposable evolution algebras

By checking all the two-dimensional **commutative** algebras that gave I. Kaygorodov and Y. Volkov for algebraically closed fields and using the procedure of simultaneous orthogonalization to recognize evolution algebras given by Y. Cabrera Casado, C. Gil Canto, D. Martín Barquero and C. Martín González (for a general field), we reached:

$\mathbf{A}_1(\frac{1}{2})$ if $\mathrm{char}(\mathbb{K}) \neq 2$ Non simple Perfect Non evolution	$\mathbf{A}_2$ if $\mathrm{char}(\mathbb{K}) = 2$ Non simple Non perfect Evolution	$\mathbf{A}_3$ Non simple Non perfect Evolution	$\mathbf{A}_4(0)$ if $\mathrm{char}(\mathbb{K}) = 2$ Simple Non evolution	$\mathbf{B}_2(\frac{1}{2})$ if $\mathrm{char}(\mathbb{K}) = 2$ Non simple Non perfect Evolution	$\mathbf{B}_3$ if $\mathrm{char}(\mathbb{K}) = 2$ Non simple Non perfect Evolution	$\mathbf{C}(\frac{1}{2}, 0)$ if $\mathrm{char}(\mathbb{K}) \neq 2$ Simple Evolution	$\mathbf{D}_1(\frac{1}{2}, 0)$ if $\mathrm{char}(\mathbb{K}) \neq 2$ Non simple Non perfect Evolution

$\mathbf{D}_2(\alpha, \alpha), \alpha \neq \frac{1}{2}$ Non simple Perfect Non evolution if $\alpha \neq 0$	$\mathbf{D}_3(\alpha, \alpha)$ if $\mathrm{char}(\mathbb{K}) = 2$ Simple if $\alpha \neq 0$ Non evolution	$\mathbf{E}_1(\alpha, \beta, \alpha, \beta)$ $4\alpha\beta \neq 1, 2\beta \neq 1, 2\alpha \neq 1$ Simple if $\alpha + \beta \neq 1$; $\alpha, \beta \neq 0$ Evolution	$\mathbf{E}_2(\frac{1}{2}, \beta, \beta)$ if $\mathrm{char}(\mathbb{K}) \neq 2$ Simple if $2\beta \neq 1, \beta \neq 0$ Evolution

$\mathbf{E}_3(\frac{1}{2}, \frac{1}{2}, \gamma), \gamma \neq 0$ if $\mathrm{char}(\mathbb{K}) \neq 2$ Simple if $\gamma \neq 1$ Non evolution	$\mathbf{E}_5(\frac{1}{2})$ if $\mathrm{char}(\mathbb{K}) \neq 2$ Not simple Perfect Non evolution

Moreover, we check if the tensor product of any two of them is an evolution algebra. In the positive case, we verify its perfectness and simpleness.

Proposition

If $\mathbb{k}$ is algebraically closed and $A_1 \otimes A_2$ is a four-dimensional simple evolution algebra, then both factors A_1 and A_2 are evolution algebras.

4-dim perfect evolution algebras and nondegeneracy

By adding $\mathrm{char}(\mathbb{k}) \neq 2$, we can give a similar result eliminating the condition of simplicity for the tensor product.

Theorem

Let A_i $(i = 1, 2)$ be $\mathbb{k}$-algebras with $\mathbb{k}$ algebraically closed and $\mathrm{char}(\mathbb{k}) \neq 2$. Assume that $A_1 \otimes A_2$ is a nontrivial $\mathbb{k}$-evolution algebra with $\dim(A_1 \otimes A_2) = 4$ and both A_i are commutative algebras. Then A_1 and A_2 are evolution algebras.

Lemma

Let A_1 and A_2 be two evolution $\mathbb{k}$-algebras. Then:

1. *A_1 and A_2 are both nondegenerate if and only if $A_1 \otimes A_2$ is nondegenerate.*
2. $\mathrm{ann}(A_1 \otimes A_2) = A_1 \otimes \mathrm{ann}(A_2) + \mathrm{ann}(A_1) \otimes A_2$.

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Perfect 4-dimensional evolution algebras

Question

When a perfect 4-dimensional evolution algebra is tensorially decomposable into two evolution algebras?

Proposition

Let A_1 and A_2 be two 2-dimensional evolution $\mathbb{k}$-algebras. If $A_1 \otimes A_2$ is a perfect evolution algebra, then either A_1 and A_2 are simple or one of them is an evolution algebra.

How do we prove that M is tensorially indecomposable?

A is a perfect 4-dimensional evolution algebra with structure matrix M with respect to a natural basis B.

Number of zeros of M

M is tensorially indecomposable if $z(M) \in \{1, 2, 3, 5, 6, 9, 11\}$ or $rank(M) = 3$ or $z_c(M) = 1$ or $z_r(M) = 1$ or $z_d(M) = 1$.

Example

$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $z(M) = 6 \Rightarrow A$ is tensorially indecomposable.

$N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $z(N) = 7, z_d(N) = 0 \Rightarrow A'$ tensorially decomposable?

In terms of the extended matrix of M

M is tensorially decomposable into two matrices if and only if $rank(Ext(M)) = 1$.

Classification of 4-dimensional perfect evolution algebras

Property

Perfect evolution algebras have, essentially, one natural basis up to permutation and multiplication by a scalar.

$$\Updownarrow$$

A perfect evolution algebra, $dim(A) = 4$, is tensorially decomposable if and only if there exist M in the orbit under the action of S_4 of one structure matrix of A such that $rank(Ext(M)) = 1$.

- M. I. Cardoso, D. Gonçalves, D. Martín Barquero, C. Martín González and M. Siles Molina. **Squares and associative representations of two dimensional evolution algebras**. J. Algebra Appl. (2020)

They determined the structure matrix of perfect evolution algebras of dimension 2:

$$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ c & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix},$$

with $1 - ab \neq 0, c \neq 0, d \neq 0$.

We make the Kronecker product of each two of them and analyze the obatined matrix.

Classification of 4-dimensional perfect evolution algebras

There are 13 different types of 4-dimensional tensorially decomposable perfect evolution algebras depending on the following set of invariants.

1. The numbers $z(M)$, $z_d(M)$ and the simplicity of the tensor product.
2. The number $z(M)$ and the characteristic polynomial $p_c(M_G)$.
3. The characteristic and minimal polynomials $p_c(M_G)$ and $p_m(M_G)$.

Family Type	$z(M_1)$	$z(M_2)$	$M_1 \otimes M_2 = M$	Simple	$z(M)$	$\begin{cases} p_c(M_G) \\ p_m(M_G) \end{cases}$
I	0	0	$\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & c \\ d & 1 \end{pmatrix} = \begin{pmatrix} 1 & c & a & ac \\ d & 1 & ad & a \\ b & bc & 1 & c \\ bd & b & d & 1 \end{pmatrix}$	Yes	0	$\begin{cases} \lambda^3(\lambda-4) \\ \lambda(\lambda-4) \end{cases}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
V	1	1	$\begin{pmatrix} 0 & 1 \\ b & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & d & a \\ b & 0 & 1 & 0 \\ bd & b & d & 1 \end{pmatrix}$	Yes	7	$\begin{cases} (\lambda^2-\lambda-1)^2 \\ (\lambda^2-\lambda-1)^2 \end{cases}$
VI	1	1	$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ bd & b & d & 1 \end{pmatrix}$	No	7	$\begin{cases} (\lambda-1)^4 \\ (\lambda-1)^3 \end{cases}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Example

Example

A evolution algebra with natural basis B and structure matrix $M = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Note that $z(M) = 7, rank(M) = 4, z_c(M) = 0, z_r(M) = 0, z_d(M) = 0$.
From these invariants we cannot say anything. So,

1. Is A perfect? $Det(M) \neq 0 \Rightarrow$Yes
2. Is A simple? It's associated graph is **not strongly connected** (we compute M^k by changing any nonzero number for 1 and there is no k such that M^k is filled with 1) $\Rightarrow$ No
3. Is A tensorially decomposable? Is there M' in the orbit of M under the action of S_4 such that $rank(Ext(M')) = 1$? $\Rightarrow$ Yes
   $M' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $Ext(M') = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix}$
4. The characteristic polynomial is $p_c(\lambda) = (\lambda - 1)^4 \Rightarrow$ type VI or X.
5. The minimal polynomial is $p_m(\lambda) = (\lambda - 1)^3 \Rightarrow$ type VI.

Thank you very much for your attention!