On the vanishing of the hyperdeterminant under certain symmetry conditions

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⊛ Summary

V vector space over a field $\mathbb{K}$ with $\mathrm{char}\mathbb{K}$ coprime with $d!$

$V^* \otimes^{(d)} \cdots \otimes V^* = \{F : V \times^{(d)} \cdots \times V \longrightarrow \mathbb{K} \text{ multilinear form}\}$

First decompose $V^* \otimes^{(d)} \cdots \otimes V^*$ into pieces (symmetry classes) W_λ according to the different partitions λ of d.

$\Rightarrow F = F_1 + F_2 + \cdots + F_k$, $F_i \in W_\lambda$

Second compute the hyperdeterminant of multilinear forms in each symmetry class.

⊛ Symmetry classes

Definition: $W \leq V^* \otimes^{(d)} \cdots \otimes V^*$ is a symmetry class if $\sigma F \in W$, $\forall F \in W$, $\forall \sigma \in S_d$.

$\sigma F(x_1, \dots, x_d) = F(x_{\sigma(1)}, \dots, x_{\sigma(d)})$, $\sigma \in S_d$.

Example: **d=2**

$$F(x_1, x_2) = \underbrace{\frac{F(x_1,x_2)+F(x_2,x_1)}{2}}_{=F_S(x_1,x_2)} + \underbrace{\frac{F(x_1,x_2)-F(x_2,x_1)}{2}}_{=F_A(x_1,x_2)}$$

$F = F_S + F_A$

$F \in V^* \otimes V^*$

$F_S \in W_{(2)}$, $F_A \in W_{(1,1)}$.

Partitions ⇝ Young diagrams

(2) ☐☐ symmetric

$(1,1)$ ☐ / ☐ skew-symmetric

$$V^* \otimes V^* = W_{(2)} \oplus W_{(1,1)} \rightsquigarrow \begin{cases} W_{(2)} = \{F : V \times V \longrightarrow \mathbb{K} \;/\; F = \sigma F, \forall \sigma \in S_2\} \\ W_{(1,1)} = \{F : V \times V \longrightarrow \mathbb{K} \;/\; F = \mathrm{sg}(\sigma)\, \sigma F, \forall \sigma \in S_2\} \end{cases}$$

Metropolis - Rota - Stein, 1991

equations for symmetry classes for $d=3$ and in general using Young diagrams

Example: $d=3$

$$F = \underbrace{\frac{1}{6}\sum_{\sigma\in S_3}\sigma F}_{=F_S} + \underbrace{\frac{1}{6}\sum \mathrm{sg}(\sigma)\sigma F}_{=F_A} + \underbrace{F - \frac{1}{3}\left(F+(123)F+(132)F\right)}_{=F_Y}$$

F_S: symmetric — (3)

F_A: skew-symmetric — $(1,1,1)$

F_Y: cyclic or Y-symmetric — $(2,1)$

STANDARD REPRESENTATION $(d-1,1)$

$$V^*\otimes V^*\otimes V^* = W_{(3)}\oplus W_{(1,1,1)}\oplus W_{(2,1)} \rightsquigarrow \begin{cases} W_{(3)} = \{F: V\times V\to \mathbb{K} \;/\; F=\sigma F,\ \forall \sigma\in S_3\} \\ W_{(1,1,1)} = \{F: V\times V\to \mathbb{K} \;/\; F=\mathrm{sg}(\sigma)\sigma F,\ \forall \sigma\in S_3\} \\ W_{(2,1)} = \{F: V\times V\to \mathbb{K} \;/\; F+(123)F+(132)F=0\} \end{cases}$$

← Metropolis - Rota - Stein, 1991

→ For a general d:

① Fulton-Harris (Representation Theory: a First Course, 1991) ⇝ Schur Functors

$$V^*\otimes^{(d)}\cdots\otimes V^* \cong \bigoplus_\lambda \left(S_\lambda V\right)^{\otimes m_\lambda}$$

$m_\lambda = \dim V_\lambda$ irreducible representation of S_d corresponding to λ.

② Enrique Arrondo (Representation Theory of Finite Groups throug (basic) Algebraic Geometry, 2021).

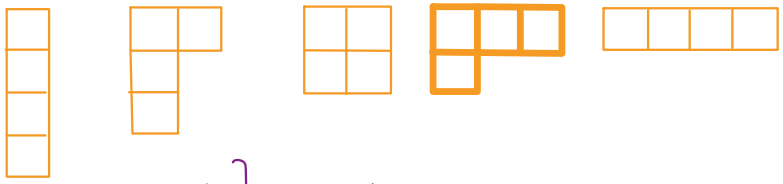
$$V^*\otimes^{(d)}\cdots\otimes V^* = \bigoplus_\lambda W_\lambda$$

sketch of this method

Example: $d = 4$

→ conjugacy classes of S_4 ⇝ $\overline{(1)}$ $\overline{(12)}$ $\overline{(12)(34)}$ $\overline{(123)}$ $\overline{(1234)}$

partitions and Young diagrams $(1,1,1,1)$ $(2,1,1)$ $(2,2)$ $(3,1)$ (4)



→ Group algebra $\mathbb{K}(S_4) = \left\{ \sum a_\sigma e_\sigma : a_\sigma \in \mathbb{K} \right\}$ with $e_\sigma e_\tau = e_{\sigma\tau}$.

→ Basis of $\mathbb{Z}\mathbb{K}(S_4)$:

- $\alpha_{(1,1,1,1)} = e_{(1)} =: \mathbf{1}$
- $\alpha_{(2,1,1)} = e_{(12)} + e_{(13)} + e_{(14)} + e_{(23)} + e_{(24)} + e_{(34)} =: \mathbf{x}$
- $\alpha_{(2,2)} = e_{(12)(34)} + (13)(24) + e_{(14)(23)} =: \mathbf{w}$
- $\alpha_{(3,1)} = e_{(123)} + e_{(132)} + e_{(124)} + e_{(142)} + e_{(134)} + e_{(143)} + e_{(234)} + e_{(243)} =: \mathbf{y}$
- $\alpha_{(4)} = e_{(1234)} + e_{(1243)} + e_{(1324)} + e_{(1342)} + e_{(1423)} + e_{(1432)} =: \mathbf{z}$

→ Multiplication in $\mathbb{Z}\mathbb{K}(S_4)$ as $\mathbb{K}$-algebra:

$x^2 = 6 + 3y + 2w$ $\quad yz = 4x + 4z$ $\quad zw = 2x + z$

$xy = 4x + 4z$ $\quad z^2 = 6 + 3y + 2w$ $\quad w^2 = 3 + 2w$

$y^2 = 8 + 4y + 8w$ $\quad xw = x + 2z$

$xz = 3y + 4w$ $\quad yw = 3y$

→ $I = \text{span}(x^2-6-3y-2w,\ xy-4x-4z,\ y^2-8-4y-8w,\ xz-3y-4w,\ yz-4x-4z,\ z^2-6-3y-2w,\ xw-x-2z,$

$yw-3y,\ zw-2x-z,\ w^2-3-2w)$

→ $V(I) = \{(6,8,6,3),\ (-6,8,-6,3),\ (0,-4,0,3),\ (-2,0,2,-1),\ (2,0,-2,-1)\}$.

→ There exists an isomorphism between $\mathbb{K}(S_4)$ and $\mathbb{K}^5$:

$$\mathbb{K}(S_4) \longrightarrow \mathbb{K}[x,y,z,w]/I \longrightarrow \mathbb{K}^5$$

$$\begin{array}{ccccc} 1 & \longmapsto & \bar{1} & \longmapsto & (1,1,1,1,1) \\ \alpha_{(2,1,1)} & \longmapsto & \bar{x} & \longmapsto & (6,-6,0,-2,2) \\ \alpha_{(3,1)} & \longmapsto & \bar{y} & \longmapsto & (8,8,-4,0,0) \\ \alpha_{(4)} & \longmapsto & \bar{z} & \longmapsto & (6,-6,0,2,-2) \\ \alpha_{(2,2)} & \longmapsto & \bar{w} & \longmapsto & (3,3,3,-1,-1) \end{array}$$

primitive idempotents ⟷ canonical basis

$\beta_S, \beta_A, \beta_1, \beta_2, \boxed{\beta_y}$ → standar representation

→ $\boxed{F = \beta_S F + \beta_A F + \beta_1 F + \beta_2 F + \beta_y F}$

→ $\boxed{V^* \otimes V^* \otimes V^* \otimes V^* = W_S \oplus W_A \oplus W_1 \oplus W_2 \oplus W_y}$ ⇝

$$\begin{cases} W_S = \beta_S\, \mathbb{K}(S_4) \\ W_A = \beta_A\, \mathbb{K}(S_4) \\ W_1 = \beta_1\, \mathbb{K}(S_4) \\ W_2 = \beta_2\, \mathbb{K}(S_4) \\ W_y = \beta_y\, \mathbb{K}(S_4) \end{cases}$$

all these equations are the same as those given by Metropolis-Rota-Stein obtained in a completely different way.

⊛ Hyperdeterminant

→ Cayley ⇝ determinants 1843 ⇝ generalization 1845, On the theory of linear transformations.

→ Gelfand, Kapranov and Zelevinski 1994 ⇝ Discriminants, Resultants and Multidimensional determinants.

↳ HYPERDETERMINANT

- Geometrically (discriminant)
- Analytically
- Algebraically

→ Ottaviani 2012 ⇝ Introduction to the hyperdeterminant and to the rank of multidimensional matrices.

↳ Definition: $\dim V = n$, $\mathbb{P}^{n-1} \overset{(d)}{\times \cdots \times} \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}(V^* \otimes \overset{(d)}{\cdots} \otimes V^*)$ Segre embedding

The hyperdeterminant of format $n \overset{(d)}{\times \cdots \times} n$ is a homogeneous polynomial function on $V^* \otimes \overset{(d)}{\cdots} \otimes V^*$ which is the defining equation of the projectively dual variety $\left(\mathbb{P}^{n-1} \overset{(d)}{\times \cdots \times} \mathbb{P}^{n-1}\right)^\vee$.

That is, $\mathrm{Det}: V^* \otimes \cdots \otimes V^* \longrightarrow \mathbb{K}$

$$F \longmapsto \mathrm{Det}(F)$$

Question: How do we compute it? Not easy :(

- explicit formulas only in some cases (boundary format case)
- elimination theory needed
- geometric interpretation

Example of computation: $2\times 2\times 2$ (GKZ)

Proposition 1.7. *The hyperdeterminant of a matrix $A = (a_{ijk})$ $(i, j, k = 0, 1)$ is given by*

$$\begin{aligned}\mathrm{Det}(A) = {} & (a_{000}^2a_{111}^2 + a_{001}^2a_{110}^2 + a_{010}^2a_{101}^2 + a_{011}^2a_{100}^2) \\ & -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} \\ & +a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) \\ & +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).\end{aligned} \tag{1.5}$$

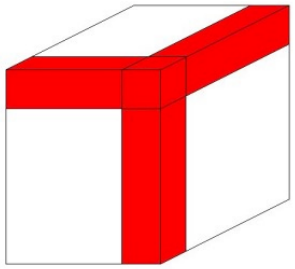
$V^*\otimes^{(d)}\cdots\otimes V^* = \{F : V\times^{(d)}\cdots\times V \longrightarrow \mathbb{K} \text{ multilinear form}\}$

Fix $\mathcal{B} = \{v_1, \dots, v_n\}$ basis of V and define

$$a_{i_1\dots i_d} = F(v_{i_1}, \dots, v_{i_d})$$

$A = (a_{i_1\dots i_d}) \Longleftrightarrow F : V\times^{(d)}\cdots\times V \longrightarrow \mathbb{K}$

Example of geometric interpretation: $n\times n\times n$ (0)



A matrix A is degenerate if and only if it becomes zero on the colored part after a linear change of coordinates.

We want to know **when $\mathrm{Det}(F) = 0$**

Property: F is degenerate $\Longleftrightarrow \mathrm{Det}(F) = 0 \Longleftrightarrow$ there exist $v_1, \dots, v_d \in V$ such that

$$F(v_1, \dots, v_{d-1}, V) = F(v_1, \dots, V, v_d) = \cdots = F(V, v_2, \dots, v_d) = 0.$$

In particular, $\mathrm{Det}(F) = 0 \Longleftrightarrow$ there exists $v \in V$ such that

$$F(v, \dots, v, V) = F(v, \dots, V, v) = \cdots = F(V, v, \dots, v) = 0.$$

With the previous property and using the equations to describe the symmetry classes,

$$V^* \overset{(d)}{\otimes} \cdots \otimes V^* = \bigoplus_{\lambda} W_\lambda \Rightarrow F = F_{\lambda_1} + F_{\lambda_2} + \cdots + F_{\lambda_K} \text{ with } F_{\lambda_k} \in W_{\lambda_k}.$$

RESULT

If $F \in W_\lambda$ with $\lambda \neq (d), (d-1,1)$, then $\mathrm{Det}(F) = 0$.

Moreover, if $F \in \bigoplus_{\lambda \neq (d),(d-1,1)} W_\lambda$ then $\mathrm{Det}(F) = 0$.

- 2016 ⇝ 3 months with Ottaviani and using Metropolis-Rota-Stein.
- 2021 ⇝ Arrondo with his method to describe symmetry classes.

No information for $d = 2$:

$W_{(2)}$ ⇝ symmetric matrices

$W_{(1,1)}$ ⇝ skew-symmetric matrices.

Example:

$(d=3)$

$$F = \underbrace{\frac{1}{6}\sum_{\sigma \in S_3} \sigma F}_{W_S} + \overbrace{\underbrace{\frac{1}{6}\sum \mathrm{sg}(\sigma)\sigma F}_{W_A}}^{\mathrm{Det}=0} + \underbrace{F - \frac{1}{3}\left(F + (123)F + (132)F\right)}_{W_Y}$$

W_S (3) W_A (1,1,1) W_Y (2,1)

$(d=4)$

$$F = \beta_S F + \overbrace{\beta_A F + \beta_1 F + \beta_2 F}^{\mathrm{Det}=0} + \beta_Y F$$

W_S (4) W_A (1,1,1,1) W_1 (2,1,1) W_2 (2,2) W_Y (3,1)

BUT

$W_{(1,1)} = W_{(d-1,1)}$

It is known that the determinant of $n \times n$ skew-symmetric matrices vanishes when n is odd

So, inside $W_{(1,1)} = W_{(d-1,1)}$ there are some pieces where the (hyp) determinant vanishes.

Improvement (2024): (Arrondo, T.)

① $W_{(d-1,1)} = \bigoplus_{i=1}^{d-1} W^i_{(d-1,1)}$ with $W^i_{(d-1,1)} = \{F \in W_{(d-1,1)} \ / \ \sigma F = \omega^i F\}$

where σ is a d-cycle and ω is primitive d-root of unity.

$F = H_1 + \cdots + H_{d-1}$ with $H_i \in W^i_{(d-1,1)}$

Sketch of the proof:

⊛ $F \in W_{(d-1,1)} \Rightarrow F + \sigma F + \sigma^2 F + \cdots + \sigma^{d-1} F = 0$

⊛ $\sum_{k=1}^{d-1} (\omega^k)^i = -1 \quad \forall i \in \{1, \ldots, d-1\}$

⊛ decomposition for $d = 4$.

$$F = \underbrace{\frac{1}{4}(F + \omega\sigma F + \omega^2\sigma^2 F + \omega^3\sigma^3 F)}_{H_3} + \underbrace{\frac{1}{4}(F + \omega^2\sigma F + (\omega^2)^2\sigma^2 F + (\omega^2)^3\sigma^3 F)}_{H_2} +$$

($\omega^4 = 1$, $\omega^6 = \omega^2$)

$$+ \underbrace{\frac{1}{4}(F + \omega^3\sigma F + (\omega^3)^2\sigma^2 F + (\omega^3)^3\sigma^3 F)}_{H_1}$$

($\omega^6 = \omega^2$, $\omega^9 = \omega$)

(2) If $F \in W^{k}_{(d-1,1)}$ with $k \in \{1,\dots,d-1\}$ then $\mathrm{Det}(F)=0$ (It does not depend on $\dim V = n$).

Sketch of the proof:

$$F \in W^{k}_{(d-1,1)} \Rightarrow \begin{cases} \sigma F = \omega^{k} F \\ F + \sigma F + \cdots + \sigma^{d-1} F = 0 \end{cases}$$

There exists $v \in V$ such that $F(v,\dots,v,V)=0$?

Since $\sigma F = \omega^{k} F \Rightarrow F(v,\dots,V,v) = \cdots = F(V,v,\dots,v) = 0 \rightsquigarrow \mathrm{Det}(F)=0$

Fix a basis of V, $B=\{v_1,\dots,v_n\}$ and take $v = \sum_{i=1}^{n} \lambda_i v_i$ $(\dim V = n)$

Since $F \notin W_{(d)} \Rightarrow F(x,\dots,x) = 0$. Hence:

$$0 = F(v,\dots,v) = F\Big(\sum_{i=1}^{n} \lambda_i v_i, \dots, \underbrace{\sum_{i=1}^{n} \lambda_i v_i}_{\text{decomposing this summatory}}\Big) =$$

$$= \lambda_1 \underbrace{F\Big(\sum_{i=1}^{n} \lambda_i v_i, \dots, \sum_{i=1}^{n} \lambda_i v_i, v_1\Big)}_{G_1(\lambda_1,\dots,\lambda_n)} + \cdots + \lambda_n \underbrace{F\Big(\sum_{i=1}^{n} \lambda_i v_i, \dots, \sum_{i=1}^{n} \lambda_i v_i, v_n\Big)}_{G_n(\lambda_1,\dots,\lambda_n)}$$

We look for non-trivial common solutions of the polynomials $G_1,\dots,G_n \in \mathbb{K}[\lambda_1,\dots,\lambda_n]_{d-1}$.

We interpret this as finding a point in $\mathbb{P}^{n-1}$ in the intersection of the corresponding n hypersurfaces. (Euler exact sequence, Chern classes, ...)

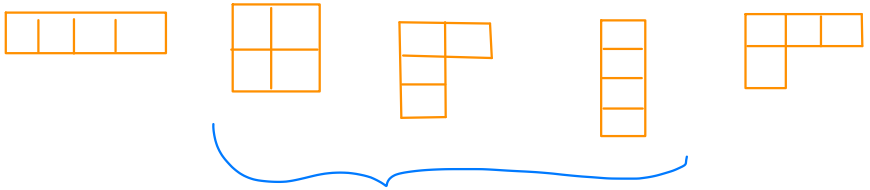
Main result:

V vector space over a field $\mathbb{K}$, fix $d \geq 3$.

Suppose $\operatorname{char}\mathbb{K}$ does not divide $d!$ and $\mathbb{K}$ contains a primitive d^{th}-root of unity ω.

If $F \in \left(\bigoplus_{\lambda \neq (d),(d-1,1)} W_\lambda\right) \oplus W_{(d-1,1)}^{\oplus i}$ for $i = 1, \dots, d-1$, then $\operatorname{Det}(F) = 0$.

$d=4$ $\quad V^* \otimes V^* \otimes V^* \otimes V^* = W_{(4)} \oplus W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}$



$\operatorname{Det}(F) = 0$

$$W_{(3,1)} = W_{(3,1)}^1 \oplus W_{(3,1)}^2 \oplus W_{(3,1)}^3$$

$\Rightarrow \operatorname{Det}(F) = 0$

$$F \in W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}^1$$
$$F \in W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}^2$$
$$F \in W_{(2,2)} \oplus W_{(2,1,1)} \oplus W_{(1,1,1,1)} \oplus W_{(3,1)}^3$$

$\Rightarrow \operatorname{Det} F = 0$

Thank you very much for your attention!